# Asymptotics of Laurent Polynomials of Odd Degree Orthogonal with Respect to Varying Exponential Weights

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**Abstract**

Let $\Lambda^{\mathbb{R}}$ denote the linear space over $\mathbb{R}$ spanned by $z^k$, $k \in \mathbb{Z}$. Define the real inner product (with varying exponential weights) $\langle \cdot, \cdot \rangle_{\mathcal{L}} : \Lambda^{\mathbb{R}} \times \Lambda^{\mathbb{R}} \to \mathbb{R}$, $(f,g) \mapsto \int_{\mathbb{R}} f(s)g(s)\exp(-\mathcal{N}\, V(s))\,\mathrm{d}s$, $\mathcal{N} \in \mathbb{N}$, where the external field $V$ satisfies: (i) $V$ is real analytic on $\mathbb{R} \setminus \{0\}$; (ii) $\lim_{|x|\to\infty}(V(x)/\ln(x^2+1)) = +\infty$; and (iii) $\lim_{|x|\to 0}(V(x)/\ln(x^{-2}+1)) = +\infty$. Orthogonalisation of the (ordered) base $\{1, z^{-1}, z, z^{-2}, z^2, \dots, z^{-k}, z^k, \dots\}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ yields the even degree and odd degree orthonormal Laurent polynomials $\{\phi_m(z)\}_{m=0}^{\infty}$: $\phi_{2n}(z) = \xi_{-n}^{(2n)} z^{-n} + \cdots + \xi_n^{(2n)} z^n$, $\xi_n^{(2n)} > 0$, and $\phi_{2n+1}(z) = \xi_{-n-1}^{(2n+1)} z^{-n-1} + \cdots + \xi_n^{(2n+1)} z^n$, $\xi_{-n-1}^{(2n+1)} > 0$. Define the even degree and odd degree monic orthogonal Laurent polynomials: $\boldsymbol{\pi}_{2n}(z) := (\xi_n^{(2n)})^{-1} \phi_{2n}(z)$ and $\boldsymbol{\pi}_{2n+1}(z) := (\xi_{-n-1}^{(2n+1)})^{-1} \phi_{2n+1}(z)$. Asymptotics in the double-scaling limit as $\mathcal{N}, n \to \infty$ such that $\mathcal{N}/n = 1 + o(1)$ of $\boldsymbol{\pi}_{2n+1}(z)$ (in the entire complex plane), $\xi_{-n-1}^{(2n+1)}$, $\phi_{2n+1}(z)$ (in the entire complex plane), and Hankel determinant ratios associated with the real-valued, bi-infinite, strong moment sequence $\left\{c_k = \int_{\mathbb{R}} s^k \exp(-\mathcal{N}\, V(s))\,\mathrm{d}s\right\}_{k \in \mathbb{Z}}$ are obtained by formulating the odd degree monic orthogonal Laurent polynomial problem as a matrix Riemann-Hilbert problem on $\mathbb{R}$, and then extracting the large-$n$ behaviour by applying the non-linear steepest-descent method introduced in [1] and further developed in [2, 3].

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# 1 Introduction

Consider the *strong Stieltjes* (resp., *strong Hamburger*) *moment problem* (SSMP) (resp., SHMP): given a doubly- or bi-infinite (moment) sequence $\{c_n\}_{n\in\mathbb{Z}}$ of real numbers:

(i) find necessary and sufficient conditions for the existence of a non-negative Borel measure $\mu_{\mathrm{MP}}^{\mathrm{SS}}$ (resp., $\mu_{\mathrm{MP}}^{\mathrm{SH}}$) on $[0,+\infty)$ (resp., $(-\infty,+\infty)$), and with infinite support, such that $c_n=\int_0^{+\infty} t^n\,\mathrm{d}\mu_{\mathrm{MP}}^{\mathrm{SS}}(t)$, $n\in\mathbb{Z}$ (resp., $c_n=\int_{-\infty}^{+\infty} t^n\,\mathrm{d}\mu_{\mathrm{MP}}^{\mathrm{SH}}(t)$, $n\in\mathbb{Z}$), where the—improper—integral is to be understood in the Riemann-Stieltjes sense;

(ii) when there is a solution, in which case the SSMP (resp., SHMP) is *determinate*, find conditions for the uniqueness of the solution; and

(iii) when there is more than one solution, in which case the SSMP (resp., SHMP) is *indeterminate*, describe the family of all solutions.

The SSMP (resp., SHMP) was introduced in 1980 (resp., 1981) by Jones *et al.* [4] (resp., Jones *et al.* [5]), and studied further in [6–10] (see, also, the review article [11]). Unlike the moment theory for the *classical Stieltjes* (resp., *classical Hamburger*) *moment problem* (SMP) [12] (resp., (HMP) [13]), wherein the theory of orthogonal polynomials [14] (and the analytic theory of continued fractions; in particular, $S$- and real $J$-fractions) enjoyed a prominent rôle (see, for example, [15]), the extension of the moment theory to the SSMP and the SHMP introduced a 'rational generalisation' of the orthogonal polynomials, namely, the *orthogonal Laurent* (or *L-*) *polynomials* (as well as the introduction of special kinds of continued fractions commonly referred to as positive-$T$ fractions), which are now introduced [6–11, 16–21]. The SHMP can also be solved using the spectral theory of unbounded self-adjoint operators in Hilbert space [22] (see, also, [23]).

For any pair $(p,q)\in\mathbb{Z}\times\mathbb{Z}$, with $p\leqslant q$, let $\Lambda_{p,q}^{\mathbb{R}}:=\left\{f:\mathbb{C}^*\to\mathbb{C};\ f(z)=\sum_{k=p}^{q}\widetilde{\lambda}_k z^k,\ \widetilde{\lambda}_k\in\mathbb{R},\ k=p,\dots,q\right\}$, where $\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$. For any $m\in\mathbb{Z}_0^+:=\{0\}\cup\mathbb{N}$, set $\Lambda_{2m}^{\mathbb{R}}:=\Lambda_{-m,m}^{\mathbb{R}}$, $\Lambda_{2m+1}^{\mathbb{R}}:=\Lambda_{-m-1,m}^{\mathbb{R}}$, and $\Lambda^{\mathbb{R}}:=\cup_{m\in\mathbb{Z}_0^+}(\Lambda_{2m}^{\mathbb{R}}\cup\Lambda_{2m+1}^{\mathbb{R}})$. (Note: the sets $\Lambda_{p,q}^{\mathbb{R}}$ and $\Lambda^{\mathbb{R}}$ form linear spaces over the field $\mathbb{R}$ with respect to the operations of addition and multiplication by a scalar.) The ordered base for $\Lambda^{\mathbb{R}}$ is $\{1,z^{-1},z,z^{-2},z^2,\dots,z^{-k},z^k,\dots\}$, corresponding to the *cyclically-repeated pole sequence* $\{\text{no pole},0,\infty,0,\infty,\dots,0,\infty,\dots\}$. For each $l\in\mathbb{Z}_0^+$ and $0\not\equiv f\in\Lambda_l^{\mathbb{R}}$, the *L-degree* of $f$, symbolically $LD(f)$, is defined as

$$LD(f):=l,$$

and the *leading coefficient* of $f$, symbolically $LC(f)$, and the *trailing coefficient* of $f$, symbolically $TC(f)$, are defined as follows:

$$LC(f):=\begin{cases}\widetilde{\lambda}_m, & l=2m,\\ \widetilde{\lambda}_{-m-1}, & l=2m+1,\end{cases}$$

and

$$TC(f):=\begin{cases}\widetilde{\lambda}_{-m}, & l=2m,\\ \widetilde{\lambda}_m, & l=2m+1.\end{cases}$$

Consider the positive measure on $\mathbb{R}$ (oriented throughout this work, unless stated otherwise, from $-\infty$ to $+\infty$) given by

$$\mathrm{d}\widetilde{\mu}(z)=\widetilde{w}(z)\mathrm{d}z,$$

with varying exponential weight function of the form

$$\widetilde{w}(z):=\exp(-\mathcal{N}\,V(z)),\quad \mathcal{N}\in\mathbb{N},$$

where the *external field* $V:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies the following conditions:

$$V \text{ is real analytic on } \mathbb{R}\setminus\{0\}; \tag{V1}$$

$$\lim_{|x|\to\infty}\Big(V(x)/\ln(x^2+1)\Big)=+\infty; \tag{V2}$$

$$\lim_{|x|\to 0}\Big(V(x)/\ln(x^{-2}+1)\Big)=+\infty. \tag{V3}$$

(For example, a rational function of the form $V(z)=\sum_{k=-2m_1}^{2m_2}\varrho_k z^k$, with $\varrho_k \in \mathbb{R}$, $k=-2m_1,\dots,2m_2$, $m_{1,2}\in\mathbb{N}$, and $\varrho_{-2m_1},\varrho_{2m_2}>0$ would suffice.) Define (uniquely) the *strong moment linear functional* $\mathcal{L}$ by its action on the basis elements of $\Lambda^{\mathbb{R}}$: $\mathcal{L}\colon \Lambda^{\mathbb{R}}\to\Lambda^{\mathbb{R}}$, $f=\sum_{k\in\mathbb{Z}}\widetilde{\lambda}_k z^k \mapsto \mathcal{L}(f):=\sum_{k\in\mathbb{Z}}\widetilde{\lambda}_k c_k$, where $c_k=\mathcal{L}(z^k)=\int_{\mathbb{R}}s^k\exp(-\mathcal{N}\,V(s))\,\mathrm{d}s$, $(k,\mathcal{N})\in\mathbb{Z}\times\mathbb{N}$. (Note that $\left\{c_k=\int_{\mathbb{R}}s^k\exp(-\mathcal{N}\,V(s))\,\mathrm{d}s,\ \mathcal{N}\in\mathbb{N}\right\}_{k\in\mathbb{Z}}$ is a bi-infinite, real-valued, *strong moment sequence*: $c_k$ is called the *kth strong moment of* $\mathcal{L}$.) Associated with the above-defined bi-infinite, real-valued, strong moment sequence $\{c_k\}_{k\in\mathbb{Z}}$ are the *Hankel determinants* $H_k^{(m)}$, $(m,k)\in\mathbb{Z}\times\mathbb{N}$ [6,7,11,17]:

$$H_0^{(m)}:=1 \qquad \text{and} \qquad H_k^{(m)}:=\begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+k-2} & c_{m+k-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+k-1} & c_{m+k} \\ c_{m+2} & c_{m+3} & \cdots & c_{m+k} & c_{m+k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-3} & c_{m+2k-2} \end{vmatrix}. \tag{1.1}$$

Define the real bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ as follows: $\langle\cdot,\cdot\rangle_{\mathcal{L}}:\Lambda^{\mathbb{R}}\times\Lambda^{\mathbb{R}}\to\mathbb{R}$, $(f,g)\mapsto\langle f,g\rangle_{\mathcal{L}}:=\mathcal{L}(f(z)g(z))=\int_{\mathbb{R}}f(s)g(s)\exp(-\mathcal{N}\,V(s))\,\mathrm{d}s$, $\mathcal{N}\in\mathbb{N}$. It is a fact [6,7,11,17] that the bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ thus defined is an inner product if and only if $H_{2m}^{(-2m)}>0$ and $H_{2m+1}^{(-2m)}>0$ for each $m\in\mathbb{Z}_0^+$ (see Equations (1.8) below, and Subsection 2.2, the proof of Lemma 2.2.2); and this fact is used, with little or no further reference, throughout this work (see, also, [24]).

**Remark 1.1.** These latter two (Hankel determinant) inequalities also appear when the question of the solvability of the SHMP is posed (in this case, the $c_k$, $k\in\mathbb{Z}$, which appear in Equations (1.1) should be replaced by $c_k^{\text{SHMP}}$, $k\in\mathbb{Z}$): indeed, if these two inequalities are true $\forall\ m\in\mathbb{Z}_0^+$, then there is a non-negative measure $\mu_{\text{MP}}^{\text{SH}}$ (on $\mathbb{R}$) with the given (real) moments. For the case of the SSMP, there are four (Hankel determinant) inequalities (in this latter case, the $c_k$, $k\in\mathbb{Z}$, which appear in Equations (1.1) should be replaced by $c_k^{\text{SSMP}}$, $k\in\mathbb{Z}$) which guarantee the existence of a non-negative measure $\mu_{\text{MP}}^{\text{SS}}$ (on $[0,+\infty)$) with the given moments, namely [4] (see, also, [6,7]): for each $m\in\mathbb{Z}_0^+$, $H_{2m}^{(-2m)}>0$, $H_{2m+1}^{(-2m)}>0$, $H_{2m}^{(-2m+1)}>0$, and $H_{2m+1}^{(-2m-1)}<0$. It is interesting to note that the former solvability conditions do not automatically imply that the positive (real) moments $\{c_k^{\text{SHMP}}\}_{k\in\mathbb{Z}_0^+}$ determine a measure via the HMP: a similar statement holds true for the SMP (see the latter four solvability conditions). ■

If $f\in\Lambda^{\mathbb{R}}$, then

$$\|f(\cdot)\|_{\mathcal{L}}:=(\langle f,f\rangle_{\mathcal{L}})^{1/2}$$

is called the *norm of f with respect to* $\mathcal{L}$: note that $\|f(\cdot)\|_{\mathcal{L}}\geqslant 0\ \forall\ f\in\Lambda^{\mathbb{R}}$, and $\|f(\cdot)\|_{\mathcal{L}}>0$ if $0\not\equiv f\in\Lambda^{\mathbb{R}}$. $\{\phi_n^{\flat}(z)\}_{n\in\mathbb{Z}_0^+}$ is called a (real) orthonormal Laurent (or *L*-) polynomial sequence (ONLPS) with respect to $\mathcal{L}$ if, $\forall\ m,n\in\mathbb{Z}_0^+$:

(i) $\phi_n^{\flat}\in\Lambda_n^{\mathbb{R}}$, that is, $LD(\phi_n^{\flat}):=n$;

(ii) $\langle\phi_m^{\flat},\phi_{n'}^{\flat}\rangle_{\mathcal{L}}=0$, $m\neq n'$, or, alternatively, $\langle f,\phi_n^{\flat}\rangle_{\mathcal{L}}=0\ \forall\ f\in\Lambda_{n-1}^{\mathbb{R}}$;

(iii) $\langle\phi_m^{\flat},\phi_m^{\flat}\rangle_{\mathcal{L}}=:\|\phi_m^{\flat}(\cdot)\|_{\mathcal{L}}^2=1$.

Orthonormalisation of $\{1,z^{-1},z,z^{-2},z^2,\dots,z^{-n},z^n,\dots\}$, corresponding to the cyclically-repeated pole sequence $\{\text{no pole},0,\infty,0,\infty,\dots,0,\infty,\dots\}$, with respect to $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ via the Gram-Schmidt orthogonalisation method, leads to the ONLPS, or, simply, orthonormal Laurent (or *L*-) polynomials (OLPs), $\{\phi_m(z)\}_{m\in\mathbb{Z}_0^+}$, which, by suitable normalisation, may be written as, for $m=2n$,

$$\phi_{2n}(z)=\xi_{-n}^{(2n)}z^{-n}+\cdots+\xi_n^{(2n)}z^n, \qquad \xi_n^{(2n)}>0, \tag{1.2}$$

and, for $m=2n+1$,

$$\phi_{2n+1}(z)=\xi_{-n-1}^{(2n+1)}z^{-n-1}+\cdots+\xi_n^{(2n+1)}z^n, \qquad \xi_{-n-1}^{(2n+1)}>0. \tag{1.3}$$

The $\phi_n$'s are normalised so that they all have real coefficients; in particular, the leading coefficients, $LC(\phi_{2n}):=\xi_n^{(2n)}$ and $LC(\phi_{2n+1}):=\xi_{-n-1}^{(2n+1)}$, $n\in\mathbb{Z}_0^+$, are both positive, $\xi_0^{(0)}=1$, and $\phi_0(z)\equiv 1$. Even though the leading coefficients $\xi_n^{(2n)}$ and $\xi_{-n-1}^{(2n+1)}$, $n\in\mathbb{Z}_0^+$, are non-zero (in particular, they are positive), no such restriction applies to the trailing coefficients, $TC(\phi_{2n}):=\xi_{-n}^{(2n)}$ and $TC(\phi_{2n+1}):=\xi_n^{(2n+1)}$, $n\in\mathbb{Z}_0^+$. Furthermore, note that, by construction:

(1) $\langle \phi_{2n}, z^{j} \rangle_{\mathcal{L}} = 0$, $j = -n, \dots, n-1$;

(2) $\langle \phi_{2n+1}, z^{j} \rangle_{\mathcal{L}} = 0$, $j = -n, \dots, n$;

(3) $\langle \phi_{j}, \phi_{k} \rangle_{\mathcal{L}} = \delta_{jk}$, $j,k \in \mathbb{Z}_{0}^{+}$, where $\delta_{jk}$ is the Kronecker delta.

It is convenient to introduce the monic orthogonal Laurent (or $L$-) polynomials, $\boldsymbol{\pi}_{j}(z)$, $j \in \mathbb{Z}_{0}^{+}$: (i) for $j = 2n$, $n \in \mathbb{Z}_{0}^{+}$, with $\boldsymbol{\pi}_{0}(z) \equiv 1$,

$$\boldsymbol{\pi}_{2n}(z) := \phi_{2n}(z)(\xi_{n}^{(2n)})^{-1} = \nu_{-n}^{(2n)} z^{-n} + \dots + z^{n}, \qquad \nu_{-n}^{(2n)} := \xi_{-n}^{(2n)} / \xi_{n}^{(2n)}; \tag{1.4}$$

and (ii) for $j = 2n+1$, $n \in \mathbb{Z}_{0}^{+}$,

$$\boldsymbol{\pi}_{2n+1}(z) := \phi_{2n+1}(z)(\xi_{-n-1}^{(2n+1)})^{-1} = z^{-n-1} + \dots + \nu_{n}^{(2n+1)} z^{n}, \qquad \nu_{n}^{(2n+1)} := \xi_{n}^{(2n+1)} / \xi_{-n-1}^{(2n+1)}. \tag{1.5}$$

The monic orthogonal $L$-polynomials, $\boldsymbol{\pi}_{j}(z)$, $j \in \mathbb{Z}_{0}^{+}$, possess the following properties:

(1) $\langle \boldsymbol{\pi}_{2n}, z^{j} \rangle_{\mathcal{L}} = 0$, $j = -n, \dots, n-1$;

(2) $\langle \boldsymbol{\pi}_{2n+1}, z^{j} \rangle_{\mathcal{L}} = 0$, $j = -n, \dots, n$;

(3) $\langle \boldsymbol{\pi}_{2n}, \boldsymbol{\pi}_{2n} \rangle_{\mathcal{L}} =: \|\boldsymbol{\pi}_{2n}(\cdot)\|_{\mathcal{L}}^{2} = (\xi_{n}^{(2n)})^{-2}$, whence $\xi_{n}^{(2n)} = 1/\|\boldsymbol{\pi}_{2n}(\cdot)\|_{\mathcal{L}}$ $(>0)$;

(4) $\langle \boldsymbol{\pi}_{2n+1}, \boldsymbol{\pi}_{2n+1} \rangle_{\mathcal{L}} =: \|\boldsymbol{\pi}_{2n+1}(\cdot)\|_{\mathcal{L}}^{2} = (\xi_{-n-1}^{(2n+1)})^{-2}$, whence $\xi_{-n-1}^{(2n+1)} = 1/\|\boldsymbol{\pi}_{2n+1}(\cdot)\|_{\mathcal{L}}$ $(>0)$.

Furthermore, in terms of the Hankel determinants, $H_{k}^{(m)}$, $(m,k) \in \mathbb{Z} \times \mathbb{N}$, associated with the real-valued, bi-infinite, strong moment sequence $\left\{ c_{k} = \int_{\mathbb{R}} s^{k} \exp(-\mathcal{N} V(s)) \, \mathrm{d}s, \ \mathcal{N} \in \mathbb{N} \right\}_{k \in \mathbb{Z}}$, the monic orthogonal $L$-polynomials, $\boldsymbol{\pi}_{j}(z)$, $j \in \mathbb{Z}_{0}^{+}$, are represented via the following determinantal formulae [6,7,11,17]: for $m \in \mathbb{Z}_{0}^{+}$,

$$\boldsymbol{\pi}_{2m}(z) = \frac{1}{H_{2m}^{(-2m)}} \begin{vmatrix} c_{-2m} & c_{-2m+1} & \cdots & c_{-1} & z^{-m} \\ c_{-2m+1} & c_{-2m+2} & \cdots & c_{0} & z^{-m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-1} & c_{0} & \cdots & c_{2m-2} & z^{m-1} \\ c_{0} & c_{1} & \cdots & c_{2m-1} & z^{m} \end{vmatrix}, \tag{1.6}$$

and

$$\boldsymbol{\pi}_{2m+1}(z) = -\frac{1}{H_{2m+1}^{(-2m)}} \begin{vmatrix} c_{-2m-1} & c_{-2m} & \cdots & c_{-1} & z^{-m-1} \\ c_{-2m} & c_{-2m+1} & \cdots & c_{0} & z^{-m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-1} & c_{0} & \cdots & c_{2m-1} & z^{m-1} \\ c_{0} & c_{1} & \cdots & c_{2m} & z^{m} \end{vmatrix}; \tag{1.7}$$

moreover, it can be shown that (see, for example, [11,17]), for $n \in \mathbb{Z}_{0}^{+}$,

$$\xi_{n}^{(2n)} \left( = \frac{1}{\|\boldsymbol{\pi}_{2n}(\cdot)\|_{\mathcal{L}}} \right) = \sqrt{\frac{H_{2n}^{(-2n)}}{H_{2n+1}^{(-2n)}}}, \qquad \xi_{-n-1}^{(2n+1)} \left( = \frac{1}{\|\boldsymbol{\pi}_{2n+1}(\cdot)\|_{\mathcal{L}}} \right) = \sqrt{\frac{H_{2n+1}^{(-2n)}}{H_{2n+2}^{(-2n-2)}}}, \tag{1.8}$$

$$\nu_{-n}^{(2n)} \left( := \frac{\xi_{-n}^{(2n)}}{\xi_{n}^{(2n)}} \right) = \frac{H_{2n}^{(-2n+1)}}{H_{2n}^{(-2n)}}, \qquad \nu_{n}^{(2n+1)} \left( := \frac{\xi_{n}^{(2n+1)}}{\xi_{-n-1}^{(2n+1)}} \right) = -\frac{H_{2n+1}^{(-2n-1)}}{H_{2n+1}^{(-2n)}}. \tag{1.9}$$

For each $m \in \mathbb{Z}_{0}^{+}$, the monic orthogonal $L$-polynomial $\boldsymbol{\pi}_{m}(z)$ and the index $m$ are called *non-singular* if $0 \neq TC(\boldsymbol{\pi}_{m}) := \begin{cases} \nu_{-n}^{(2n)}, & m = 2n, \\ \nu_{n}^{(2n+1)}, & m = 2n+1; \end{cases}$ otherwise, $\boldsymbol{\pi}_{m}(z)$ and $m$ are *singular*. From Equations (1.9), it can be seen that, for each $m \in \mathbb{Z}_{0}^{+}$:

(i) $\boldsymbol{\pi}_{2m}(z)$ is non-singular (resp., singular) if $H_{2m}^{(-2m+1)} \neq 0$ (resp., $H_{2m}^{(-2m+1)} = 0$);

(ii) $\boldsymbol{\pi}_{2m+1}(z)$ is non-singular (resp., singular) if $H^{(-2m-1)}_{2m+1} \neq 0$ (resp., $H^{(-2m-1)}_{2m+1} = 0$).

For each $m \in \mathbb{Z}^{+}_{0}$, let $\mu_{2m} := \mathrm{card}\{z;\ \boldsymbol{\pi}_{2m}(z)=0\}$, and $\mu_{2m+1} := \mathrm{card}\{z; \boldsymbol{\pi}_{2m+1}(z)=0\}$. It is an established fact [6, 7, 17] that, for $m \in \mathbb{Z}^{+}_{0}$:

(1) the zeros of $\boldsymbol{\pi}_{2m}(z)$ are real, simple, and non-zero, and $\mu_{2m} = 2m$ (resp., $2m-1$) if $\boldsymbol{\pi}_{2m}(z)$ is non-singular (resp., singular);

(2) the zeros of $\boldsymbol{\pi}_{2m+1}(z)$ are real, simple, and non-zero, and $\mu_{2m+1} = 2m+1$ (resp., $2m$) if $\boldsymbol{\pi}_{2m+1}(z)$ is non-singular (resp., singular).

For each $m \in \mathbb{Z}^{+}_{0}$, it can be shown that, via a straightforward factorisation argument and using Equations (1.6) and (1.7):

(i) if $\boldsymbol{\pi}_{2m}(z)$ is non-singular, upon setting $\left\{\alpha^{(2m)}_{k},\ k=1,\dots,2m\right\} := \{z;\ \boldsymbol{\pi}_{2m}(z)=0\}$,

$$\prod_{k=1}^{2m} \alpha^{(2m)}_{k} = \nu^{(2m)}_{-m};$$

(ii) if $\boldsymbol{\pi}_{2m+1}(z)$ is non-singular, upon setting $\left\{\alpha^{(2m+1)}_{k},\ k=1,\dots,2m+1\right\} := \{z;\ \boldsymbol{\pi}_{2m+1}(z)=0\}$,

$$\prod_{k=1}^{2m+1} \alpha^{(2m+1)}_{k} = -\left(\nu^{(2m+1)}_{m}\right)^{-1}.$$

**Remark 1.2.** It is important to note [10] that the classical and strong moment problems (SMP, HMP, SSMP, and SHMP) are special cases of a more general theory, where moments corresponding to an arbitrary, countable sequence of (fixed) points are involved (in the classical and strong moment cases, respectively, the points are $\infty$ repeated and $0, \infty$ cyclically repeated), and where *orthogonal rational functions* [25] play the rôle of orthogonal polynomials and orthogonal Laurent (or $L$-) polynomials; furthermore, since $L$-polynomials are rational functions with (fixed) poles at the origin and at the point at infinity, the step towards a more general theory where poles are at arbitrary, but fixed, positions/locations in $\mathbb{C} \cup \{\infty\}$ is natural. ■

Unlike orthogonal polynomials, which satisfy a system of three-term recurrence relations, monic orthogonal, and orthonormal, $L$-polynomials may satisfy recurrence relations consisting of a pair of four-term recurrence relations [11], a pair of systems of three- or five-term recurrence relations (which is guaranteed in the case when the corresponding monic orthogonal, and orthonormal, $L$-polynomials are non-singular) [11, 16, 17], or a system consisting of four five-term recurrence relations [23].

**Remark 1.3.** The non-vanishing of the leading and trailing coefficients of the OLPs $\{\phi_m(z)\}_{m=0}^{\infty}$, that is,

$$LC(\phi_m) := \begin{cases} \xi^{(2n)}_{n}, & m=2n, \\ \xi^{(2n+1)}_{-n-1}, & m=2n+1, \end{cases}$$

and

$$TC(\phi_m) := \begin{cases} \xi^{(2n)}_{-n}, & m=2n, \\ \xi^{(2n+1)}_{n}, & m=2n+1, \end{cases}$$

respectively, is of paramount importance: if both these conditions are not satisfied, then the 'length' of the recurrence relations may be greater than three [16] (see, also, [24]). ■

It can be shown that (see, for example, [17]; see, also, Chapter 11 of [25]), if $\{\boldsymbol{\pi}_m(z)\}_{m \in \mathbb{Z}^{+}_{0}}$, as defined above, is a non-singular, monic orthogonal $L$-polynomial sequence, that is, $H^{(-2n+1)}_{2n} \neq 0$ $(m=2n)$ and $H^{(-2n-1)}_{2n+1} \neq 0$ $(m=2n+1)$, then $\{\boldsymbol{\pi}_m(z)\}_{m \in \mathbb{Z}^{+}_{0}}$ satisfy the pair of three-term recurrence relations

$$\boldsymbol{\pi}_{2m+1}(z) = \left(\frac{z^{-1}}{\beta^{\natural}_{2m}} + \beta^{\natural}_{2m+1}\right)\boldsymbol{\pi}_{2m}(z) + \lambda^{\natural}_{2m+1}\boldsymbol{\pi}_{2m-1}(z),$$

$$\boldsymbol{\pi}_{2m+2}(z)=\left(\frac{z}{\beta^{\natural}_{2m+1}}+\beta^{\natural}_{2m+2}\right)\boldsymbol{\pi}_{2m+1}(z)+\lambda^{\natural}_{2m+2}\boldsymbol{\pi}_{2m}(z),$$

where $\boldsymbol{\pi}_{-1}(z)\equiv 0$,

$$\beta^{\natural}_{2m}=\nu^{(2m)}_{-m}, \qquad \beta^{\natural}_{2m+1}=\nu^{(2m+1)}_{m},$$
$$\lambda^{\natural}_{2m+1}=-\frac{H^{(-2m-1)}_{2m+1}H^{(-2m+2)}_{2m-1}}{H^{(-2m)}_{2m}H^{(-2m+1)}_{2m}} \quad (\neq 0), \qquad \lambda^{\natural}_{2m+2}=-\frac{H^{(-2m-1)}_{2m+2}H^{(-2m)}_{2m}}{H^{(-2m)}_{2m+1}H^{(-2m-1)}_{2m+1}} \quad (\neq 0),$$

and $\lambda_j\beta_{j-1}/\beta_j>0$, $j\in\mathbb{N}$, with $\lambda_1:=-c_{-1}$, leading to a *tri-diagonal-type Laurent-Jacobi matrix* $\mathcal{F}$ for the 'mixed' mapping

$$\mathfrak{F}\colon \Lambda^{\mathbb{R}}\to\Lambda^{\mathbb{R}},\ \ f(z)\mapsto (z^{-1}(\oplus_{n=0}^{\infty}\operatorname{diag}(1,0))+z(\oplus_{n=0}^{\infty}\operatorname{diag}(0,1)))f(z),$$

where $\oplus_{n=0}^{\infty}\operatorname{diag}(1,0):=\operatorname{diag}(1,0,\ldots,1,0,\ldots)$ and $\oplus_{n=0}^{\infty}\operatorname{diag}(0,1):=\operatorname{diag}(0,1,\ldots,0,1,\ldots)$,

$$\mathcal{F}=\operatorname{diag}\left(\beta^{\natural}_0,\beta^{\natural}_1,\beta^{\natural}_2,\ldots\right)\begin{pmatrix}
-\beta^{\natural}_1 & 1 & & & & & & & \\
-\lambda^{\natural}_2 & -\beta^{\natural}_2 & 1 & & & & & & \\
 & -\lambda^{\natural}_3 & -\beta^{\natural}_3 & 1 & & & & & \\
 & & -\lambda^{\natural}_4 & -\beta^{\natural}_4 & 1 & & & & \\
 & & & -\lambda^{\natural}_5 & -\beta^{\natural}_5 & 1 & & & \\
 & & & & -\lambda^{\natural}_6 & -\beta^{\natural}_6 & 1 & & \\
 & & & & \ddots & \ddots & \ddots & & \\
 & & & & & -\lambda^{\natural}_{2m+1} & -\beta^{\natural}_{2m+1} & 1 & \\
 & & & & & & -\lambda^{\natural}_{2m+2} & -\beta^{\natural}_{2m+2} & 1 \\
 & & & & & & & \ddots & \ddots\ \ddots
\end{pmatrix},$$

with zeros outside the indicated diagonals (in terms of $\{\phi_m(z)\}_{m\in\mathbb{Z}_0^+}$, the pair of three-term recurrence relations reads [16]:

$$\phi_{2m+1}(z)=(z^{-1}+\mathfrak{g}_{2m+1})\phi_{2m}(z)+\mathfrak{f}_{2m+1}\phi_{2m-1}(z),$$
$$\phi_{2m+2}(z)=(1+\mathfrak{g}_{2m+2}z)\phi_{2m+1}(z)+\mathfrak{f}_{2m+2}\phi_{2m}(z),$$

where $\mathfrak{f}_{2m+1},\mathfrak{f}_{2m+2}\neq 0$, $m\in\mathbb{Z}_0^+$, $\phi_{-1}(z)\equiv 0$, and $\phi_0(z)\equiv 1$); otherwise, $\{\boldsymbol{\pi}_m(z)\}_{m\in\mathbb{Z}_0^+}$ satisfy the following pair of five-term recurrence relations [17], with $\boldsymbol{\pi}_{-j}(z)\equiv 0$, $j=1,2$,

$$\begin{aligned}\boldsymbol{\pi}_{2m+2}(z)&=\gamma^{\flat}_{2m+2,2m-2}\boldsymbol{\pi}_{2m-2}(z)+\gamma^{\flat}_{2m+2,2m-1}\boldsymbol{\pi}_{2m-1}(z)+(z+\gamma^{\flat}_{2m+2,2m})\boldsymbol{\pi}_{2m}(z)\\&\quad+\gamma^{\flat}_{2m+2,2m+1}\boldsymbol{\pi}_{2m+1}(z),\\
\boldsymbol{\pi}_{2m+3}(z)&=\gamma^{\flat}_{2m+3,2m-1}\boldsymbol{\pi}_{2m-1}(z)+\gamma^{\flat}_{2m+3,2m}\boldsymbol{\pi}_{2m}(z)+(z^{-1}+\gamma^{\flat}_{2m+3,2m+1})\boldsymbol{\pi}_{2m+1}(z)\\&\quad+\gamma^{\flat}_{2m+3,2m+2}\boldsymbol{\pi}_{2m+2}(z),\end{aligned}$$

where $\gamma_{l,k}=0$, $k<0$, $l\geqslant 2$, leading to a *penta-diagonal-type Laurent-Jacobi matrix* $\mathcal{G}$ for the 'mixed' mapping

$$\mathfrak{G}\colon \Lambda^{\mathbb{R}}\to\Lambda^{\mathbb{R}},\ \ g(z)\mapsto (z(\oplus_{n=0}^{\infty}\operatorname{diag}(1,0))+z^{-1}(\oplus_{n=0}^{\infty}\operatorname{diag}(0,1)))g(z),$$

$$\mathcal{G}=\begin{pmatrix}
-\gamma^{\flat}_{2,0} & -\gamma^{\flat}_{2,1} & 1 & & & & & & & \\
-\gamma^{\flat}_{3,0} & -\gamma^{\flat}_{3,1} & -\gamma^{\flat}_{3,2} & 1 & & & & & & \\
-\gamma^{\flat}_{4,0} & -\gamma^{\flat}_{4,1} & -\gamma^{\flat}_{4,2} & -\gamma^{\flat}_{4,3} & 1 & & & & & \\
 & -\gamma^{\flat}_{5,1} & -\gamma^{\flat}_{5,2} & -\gamma^{\flat}_{5,3} & -\gamma^{\flat}_{5,4} & 1 & & & & \\
 & & -\gamma^{\flat}_{6,2} & -\gamma^{\flat}_{6,3} & -\gamma^{\flat}_{6,4} & -\gamma^{\flat}_{6,5} & 1 & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & -\gamma^{\flat}_{2m+2,2m-2} & -\gamma^{\flat}_{2m+2,2m-1} & -\gamma^{\flat}_{2m+2,2m} & -\gamma^{\flat}_{2m+2,2m+1} & 1 & & \\
 & & & & -\gamma^{\flat}_{2m+3,2m-1} & -\gamma^{\flat}_{2m+3,2m} & -\gamma^{\flat}_{2m+3,2m+1} & -\gamma^{\flat}_{2m+3,2m+2} & 1 & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},$$

with zeros outside the indicated diagonals. The general form of these (system of) recurrence relations is a pair of three- and five-term recurrence relations [23]: for $n \in \mathbb{Z}_0^+$,

$$z\phi_{2n+1}(z) = b_{2n+1}^{\sharp}\phi_{2n}(z) + a_{2n+1}^{\sharp}\phi_{2n+1}(z) + b_{2n+2}^{\sharp}\phi_{2n+2}(z),$$
$$z\phi_{2n}(z) = c_{2n}^{\sharp}\phi_{2n-2}(z) + b_{2n}^{\sharp}\phi_{2n-1}(z) + a_{2n}^{\sharp}\phi_{2n}(z) + b_{2n+1}^{\sharp}\phi_{2n+1}(z) + c_{2n+2}^{\sharp}\phi_{2n+2}(z),$$

where all the coefficients are real, $c_0^{\sharp} = b_0^{\sharp} = 0$, and $c_{2k}^{\sharp} > 0$, $k \in \mathbb{N}$, and

$$z^{-1}\phi_{2n}(z) = \beta_{2n}^{\sharp}\phi_{2n-1}(z) + \alpha_{2n}^{\sharp}\phi_{2n}(z) + \beta_{2n+1}^{\sharp}\phi_{2n+1}(z),$$
$$z^{-1}\phi_{2n+1}(z) = \gamma_{2n+1}^{\sharp}\phi_{2n-1}(z) + \beta_{2n+1}^{\sharp}\phi_{2n}(z) + \alpha_{2n+1}^{\sharp}\phi_{2n+1}(z) + \beta_{2n+2}^{\sharp}\phi_{2n+2}(z) + \gamma_{2n+3}^{\sharp}\phi_{2n+3}(z),$$

where all the coefficients are real, $\beta_0^{\sharp} = \gamma_1^{\sharp} = 0$, $\beta_1^{\sharp} > 0$, and $\gamma_{2l+1}^{\sharp} > 0$, $l \in \mathbb{N}$, leading, respectively, to the real-symmetric, *tri-penta-diagonal-type Laurent-Jacobi matrices*, $\mathcal{J}$ and $\mathcal{K}$, for the mappings

$$\mathcal{J}\colon \Lambda^{\mathbb{R}} \to \Lambda^{\mathbb{R}},\ j(z) \mapsto zj(z) \qquad \text{and} \qquad \mathcal{K}\colon \Lambda^{\mathbb{R}} \to \Lambda^{\mathbb{R}},\ k(z) \mapsto z^{-1}k(z),$$

$$\mathcal{J} = \left(\begin{array}{ccccccccccccc}
a_0^{\sharp} & b_1^{\sharp} & c_2^{\sharp} & & & & & & & & & & \\
b_1^{\sharp} & a_1^{\sharp} & b_2^{\sharp} & & & & & & & & & & \\
c_2^{\sharp} & b_2^{\sharp} & a_2^{\sharp} & b_3^{\sharp} & c_4^{\sharp} & & & & & & & & \\
 & & b_3^{\sharp} & a_3^{\sharp} & b_4^{\sharp} & & & & & & & & \\
 & & c_4^{\sharp} & b_4^{\sharp} & a_4^{\sharp} & b_5^{\sharp} & c_6^{\sharp} & & & & & & \\
 & & & & b_5^{\sharp} & a_5^{\sharp} & b_6^{\sharp} & & & & & & \\
 & & & & c_6^{\sharp} & b_6^{\sharp} & a_6^{\sharp} & b_7^{\sharp} & c_8^{\sharp} & & & & \\
 & & & & & & b_7^{\sharp} & a_7^{\sharp} & b_8^{\sharp} & & & & \\
 & & & & & & c_8^{\sharp} & b_8^{\sharp} & a_8^{\sharp} & b_9^{\sharp} & c_{10}^{\sharp} & & \\
 & & & & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & & & b_{2k+1}^{\sharp} & a_{2k+1}^{\sharp} & b_{2k+2}^{\sharp} & & \\
 & & & & & & & & c_{2k+2}^{\sharp} & b_{2k+2}^{\sharp} & a_{2k+2}^{\sharp} & b_{2k+3}^{\sharp} & c_{2k+4}^{\sharp} \\
 & & & & & & & & & & \ddots & \ddots & \ddots
\end{array}\right),$$

and

$$\mathcal{K} = \left(\begin{array}{ccccccccccccc}
\alpha_0^{\sharp} & \beta_1^{\sharp} & & & & & & & & & & & \\
\beta_1^{\sharp} & \alpha_1^{\sharp} & \beta_2^{\sharp} & \gamma_3^{\sharp} & & & & & & & & & \\
 & \beta_2^{\sharp} & \alpha_2^{\sharp} & \beta_3^{\sharp} & & & & & & & & & \\
 & \gamma_3^{\sharp} & \beta_3^{\sharp} & \alpha_3^{\sharp} & \beta_4^{\sharp} & \gamma_5^{\sharp} & & & & & & & \\
 & & & \beta_4^{\sharp} & \alpha_4^{\sharp} & \beta_5^{\sharp} & & & & & & & \\
 & & & \gamma_5^{\sharp} & \beta_5^{\sharp} & \alpha_5^{\sharp} & \beta_6^{\sharp} & \gamma_7^{\sharp} & & & & & \\
 & & & & & \beta_6^{\sharp} & \alpha_6^{\sharp} & \beta_7^{\sharp} & & & & & \\
 & & & & & \gamma_7^{\sharp} & \beta_7^{\sharp} & \alpha_7^{\sharp} & \beta_8^{\sharp} & \gamma_9^{\sharp} & & & \\
 & & & & & & \beta_8^{\sharp} & \alpha_8^{\sharp} & \beta_9^{\sharp} & & & & \\
 & & & & & & \ddots & \ddots & \ddots & & & & \\
 & & & & & & & & \gamma_{2k+1}^{\sharp} & \beta_{2k+1}^{\sharp} & \alpha_{2k+1}^{\sharp} & \beta_{2k+2}^{\sharp} & \gamma_{2k+3}^{\sharp} \\
 & & & & & & & & & & \beta_{2k+2}^{\sharp} & \alpha_{2k+2}^{\sharp} & \beta_{2k+3}^{\sharp} \\
 & & & & & & & & & & \ddots & \ddots & \ddots
\end{array}\right),$$

with zeros outside the indicated diagonals; moreover, as shown in [23], $\mathcal{J}$ and $\mathcal{K}$ are formal inverses, that is, $\mathcal{J}\mathcal{K} = \mathcal{K}\mathcal{J} = \mathrm{diag}(1, \ldots, 1, \ldots)$ (see, also, [26–30]). (Note: $\mathcal{J}$ (resp., $\mathcal{K}$) is the matrix representation of the multiplication (resp., inversion) operator in the real linear space of rational functions $\mathbb{R}[z, z^{-1}]$ when $L$-polynomials are chosen as basis.)

It is worth mentioning that a subset of the multitudinous applications $L$-polynomials and their associated Laurent-Jacobi matrices find in numerical analysis (quadrature formulae) and trigonometric moment problems [31], the spectral theory of self-adjoint operators in infinite-dimensional (necessarily separable) Hilbert spaces [22, 23], complex approximation theory (two-point Padé approximants) [32–34], the direct/inverse scattering theory for the (finite) relativistic Toda lattice [35] (see, also, [36]), and the (classical) Pick-Nevanlinna problem [37] are discussed in Section 1 of [38] (see, also, [25], and the references therein). It turns out that, as a recurring theme, $n \to \infty$ asymptotics of $L$-polynomials are an essential calculational ingredient in analyses related to the above-mentioned, seemingly disparate, topics.

Now that the principal objects have been defined, namely, the monic OLPs, $\{\boldsymbol{\pi}_m(z)\}_{m \in \mathbb{Z}_0^+}$, and OLPs, $\{\phi_m(z)\}_{m \in \mathbb{Z}_0^+}$, it's time to state that the purpose of the present (three-fold) series of works, of which the present article constitutes Part II, is to analyse the behaviour in the double-scaling limit as $\mathcal{N}, n \to \infty$ such that $z_o := \mathcal{N}/n = 1 + o(1)$ (the simplified 'notation' $n \to \infty$ will be adopted) of the $L$-polynomials $\boldsymbol{\pi}_n(z)$ and $\phi_n(z)$ in $\mathbb{C}$, orthogonal with respect to the varying exponential measure[1] $\mathrm{d}\mu(z) = \exp(-n\widetilde{V}(z))\,\mathrm{d}z$, where $\widetilde{V}(z) := z_o V(z)$, and the 'scaled' external field[2] $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfies conditions (2.3)–(2.5) (see Subsection 2.2), as well as of the associated norming constants and coefficients of the (system of) recurrence relations; more precisely, then:

**(i)** in this work (Part II), asymptotics (as $n \to \infty$) of $\boldsymbol{\pi}_{2n+1}(z)$ (in the entire complex plane) and $\xi^{(2n+1)}_{-n-1}$, thus $\phi_{2n+1}(z)$ (cf. Equation (1.5)) and the Hankel determinant ratio $H^{(-2n)}_{2n+1}/H^{(-2n-2)}_{2n+2}$ (cf. Equations (1.8)), are obtained;

**(ii)** in the previous work [38] (Part I), asymptotics (as $n \to \infty$) of $\boldsymbol{\pi}_{2n}(z)$ (in the entire complex plane) and $\xi^{(2n)}_{n}$, thus $\phi_{2n}(z)$ (cf. Equation (1.4)) and the Hankel determinant ratio $H^{(-2n)}_{2n}/H^{(-2n)}_{2n+1}$ (cf. Equations (1.8)), were obtained;

**(iii)** in Part III [40], asymptotics (as $n \to \infty$) of $\nu^{(2n)}_{-n}$ $(= H^{(-2n+1)}_{2n}/H^{(-2n)}_{2n})$ and $\xi^{(2n)}_{-n}$, $\nu^{(2n+1)}_{n}$ $(= -H^{(-2n-1)}_{2n+1}/H^{(-2n)}_{2n+1})$ and $\xi^{(2n+1)}_{n}$, $\prod_{k=1}^{2n} \alpha^{(2n)}_{k}$ $(= \nu^{(2n)}_{-n})$, and $\prod_{k=1}^{2n+1} \alpha^{(2n+1)}_{k}$ $(= -(\nu^{(2n+1)}_{n})^{-1})$, as well as of the (elements of the) Laurent-Jacobi matrices, $\mathcal{J}$ and $\mathcal{K}$, and other, related, quantities constructed from the coefficients of the three- and five-term recurrence relations, are obtained.

The above-mentioned asymptotics (as $n \to \infty$) are obtained by reformulating, *à la* Fokas-Its-Kitaev [41, 42], the corresponding 'even degree' and 'odd degree' monic $L$-polynomial problems as (matrix) Riemann-Hilbert problems (RHPs) on $\mathbb{R}$, and then studying the large-$n$ behaviour of the corresponding solutions. The paradigm for the asymptotic (as $n \to \infty$) analysis of the respective (matrix) RHPs is a union of the Deift-Zhou (DZ) non-linear steepest-descent method [1, 2], used for the asymptotic analysis of undulatory RHPs, and the extension of Deift-Venakides-Zhou [3], incorporating into the DZ method a non-linear analogue of the WKB method, making the asymptotic analysis of fully non-linear problems tractable (it should be mentioned that, in this context, the equilibrium measure [43] plays an absolutely crucial rôle in the analysis [44]); see, also, the multitudinous extensions and applications of the DZ method [45–69]. It is worth mentioning that asymptotics for Laurent-type polynomials and their zeros have been obtained in [33, 70] (see, also, [71–73]).

Unlike the large-$n$ asymptotic analysis for the orthogonal polynomials case, which is related to one (matrix) RHP normalised at the point at infinity, the large-$n$ asymptotic analysis for the OLPs requires the consideration of two different families of (matrix) RHPs, one for even degree (see Subsection 2.2, **RHP1** and Lemma 2.2.1), and one for odd degree (see Subsection 2.2, **RHP2** and Lemma 2.2.2): **RHP1** is normalised at the point at infinity, whereas **RHP2** is normalised at $z = 0$. The technical details, therefore, related to the large-$n$ asymptotic analyses of **RHP1** and **RHP2** are different, and must be carried through independently. Further, important, albeit technical, distinctions are:

- the associated $g$-functions are different (see [38], Equation (2.12), for the even degree case, and Equation (2.13) of the present article for the odd degree case);

---

[1] Note that $LD(\boldsymbol{\pi}_m) = LD(\phi_m) = \begin{cases} 2n, & m = \text{even}, \\ 2n+1, & m = \text{odd}, \end{cases}$ coincides with the parameter in the measure of orthogonality: the large parameter, $n$, enters simultaneously into the $L$-degree of the $L$-polynomials and the (varying exponential) weight; thus, asymptotics of the $L$-polynomials are studied along a 'diagonal strip' of a doubly-indexed sequence.

[2] For real non-analytic external fields, see the recent work [39].

- the respective variational problems are different, which means that the supports of the associated equilibrium measures are different (see [38], Lemmas 3.1–3.3, for the even degree case, and Lemmas 3.1–3.3 of the present article for the odd degree case);
- even though the supports consist of the union of a finite number of compact real intervals, the systems of transcendental equations (finite in number) which characterise the end-points of the supports of the respective equilibrium measures are different (see [38], Lemmas 3.5 and 3.6, for the even degree case, and Lemmas 3.5 and 3.6 of the present article for the odd degree case);
- the associated 'small-norm' RHPs, due to the difference in normalisations, are different, which gives rise to the appearance of several new—residue—terms in the solution of the odd degree small-norm RHP, which do not arise in the solution of the even degree small-norm RHP (see, in particular, [38], Lemmas 4.8 and 5.2, for the even degree case, and Lemmas 4.8 and 5.2 of the present article for the odd degree case); and
- certain error analyses are more complicated for the odd degree case, because the difference in normalisations requires that twice as many matrix-operator norms be estimated for the odd degree case, which makes the asymptotic analysis more tedious and involved (see, in particular, [38], Proposition 5.2 and Lemma 5.2, for the even degree case, and Proposition 5.2 and Lemma 5.2 of the present article for the odd degree case).

For these reasons, as well other, technical ones, attempting to meld together the even degree and odd degree cases into one article would hamper the readability of an already lengthy analysis. So, despite the repeated, over-arching scheme of analysis, the large-$n$ asymptotic behaviour for the even degree OLPs (and related quantities) is studied in [38], and the large-$n$ asymptotic behaviour for the odd degree OLPs (and related quantities) is the subject of the present article.

This article is organised as follows. In Section 2, necessary facts from the theory of (compact) Riemann surfaces are given, the respective 'even degree' and 'odd degree' RHPs on $\mathbb{R}$ are stated and the corresponding variational problems for the associated equilibrium measures are discussed, and the main results of this work, namely, asymptotics (as $n \to \infty$) of $\boldsymbol{\pi}_{2n+1}(z)$ (in $\mathbb{C}$), and $\xi^{(2n+1)}_{-n-1}$ and $\phi_{2n+1}(z)$ (in $\mathbb{C}$) are stated in Theorems 2.3.1 and 2.3.2, respectively. In Section 3, the detailed analysis of the 'odd degree' variational problem and the associated equilibrium measure is undertaken, including the construction of the so-called $g$-function, and the RHP formulated in Section 2 is reformulated as an equivalent, auxiliary RHP, which, in Sections 4 and 5, is augmented, by means of a sequence of contour deformations and transformations *à la* Deift-Venakides-Zhou, into simpler, 'model' (matrix) RHPs which, as $n \to \infty$, and in conjunction with the Beals-Coifman construction [74] (see, also, the extension of Zhou [75]) for the integral representation of the solution of a matrix RHP on an oriented contour, are solved explicitly (in closed form) in terms of Riemann theta functions (associated with the underlying finite-genus hyperelliptic Riemann surface) and Airy functions, from which the final asymptotic (as $n \to \infty$) results stated in Theorems 2.3.1 and 2.3.2 are proved. The paper concludes with an Appendix.

**Remark 1.4.** The even degree OLPs, $\phi_{2n}(z), n \in \mathbb{Z}_0^+$, are related, in a way, to the polynomials orthogonal with respect to the varying weight $\widehat{w}(z) := z^{-2n} \exp(-\mathcal{N}\, V(z))$, $\mathcal{N} \in \mathbb{N}$: this follows directly from the orthogonality relation satisfied by $\phi_{2n}(z)$. This does not help with any of the algebraic relations, such as the system of three- and five-term recurrence relations; however, this does provide for an alternative approach to computing large-$n$ asymptotics for $\phi_{2n}(z)$. The connection is not so clear for the odd degree OLPs, $\phi_{2n+1}(z)$, $n \in \mathbb{Z}_0^+$. Indeed, in this latter case, the associated (density of the) measure for the orthogonal polynomials would take the form $\mathrm{d}\widehat{\mu}(z) := z^{-2n-1} \exp(-\mathcal{N}\, V(z))\, \mathrm{d}z$, and this measure changes signs, which causes a number of difficulties in the large-$n$ asymptotic analysis. In this paper, these connections are not used, and a complete asymptotic analysis of the odd degree OLPs is carried out, directly. ∎

# 2 Hyperelliptic Riemann Surfaces, The Riemann-Hilbert Problems, and Summary of Results

In this section, necessary facts from the theory of hyperelliptic Riemann surfaces are given (see Subsection 2.1), the respective RHPs on $\mathbb{R}$ for the even degree and the odd degree monic orthogonal $L$-polynomials are formulated and the corresponding variational problems for the associated equilibrium measures are discussed (see Subsection 2.2), and the asymptotics (as $n \to \infty$) for $\boldsymbol{\pi}_{2n+1}(z)$

(in the entire complex plane), and $\xi_{-n-1}^{(2n+1)}$ and $\phi_{2n+1}(z)$ (in the entire complex plane) are given in Theorems 2.3.1 and 2.3.2, respectively (see Subsection 2.3).

Before proceeding, however, the notation/nomenclature used throughout this work is summarised.

**Notational Conventions**

(1) $\mathrm{I}=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ is the $2\times 2$ identity matrix, $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma_2=\left(\begin{smallmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{smallmatrix}\right)$ and $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ are the Pauli matrices, $\sigma_+=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\sigma_-=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ are, respectively, the raising and lowering matrices, $\mathbf{0}=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, $\mathbb{R}_\pm:=\{x\in\mathbb{R};\ \pm x>0\}$, $\mathbb{C}_\pm:=\{z\in\mathbb{C};\ \pm\mathrm{Im}(z)>0\}$, and $\mathrm{sgn}(x):=0$ if $x=0$ and $x|x|^{-1}$ if $x\neq 0$;

(2) for a scalar $\omega$ and a $2\times2$ matrix $\Upsilon$, $\omega^{\mathrm{ad}(\sigma_3)}\Upsilon:=\omega^{\sigma_3}\Upsilon\omega^{-\sigma_3}$;

(3) a contour $\mathcal{D}$ which is the finite union of piecewise-smooth, simple curves (as closed sets) is said to be *orientable* if its complement $\mathbb{C}\setminus\mathcal{D}$ can always be divided into two, possibly disconnected, disjoint open sets $\mho^+$ and $\mho^-$, either of which has finitely many components, such that $\mathcal{D}$ admits an orientation so that it can either be viewed as a positively oriented boundary $\mathcal{D}^+$ for $\mho^+$ or as a negatively oriented boundary $\mathcal{D}^-$ for $\mho^-$ [75], that is, the (possibly disconnected) components of $\mathbb{C}\setminus\mathcal{D}$ can be coloured by $+$ or $-$ in such a way that the $+$ regions do not share boundary with the $-$ regions, except, possibly, at finitely many points [76];

(4) for each segment of an oriented contour $\mathcal{D}$, according to the given orientation, the "+" side is to the left and the "-" side is to the right as one traverses the contour in the direction of orientation, that is, for a matrix $\mathcal{A}_{ij}(z)$, $i,j=1,2$, $(\mathcal{A}_{ij}(z))_\pm$ denote the non-tangential limits $(\mathcal{A}_{ij}(z))_\pm:=\lim_{\substack{z'\to z\\ z'\,\in\,\pm\,\text{side of}\,\mathcal{D}}}\mathcal{A}_{ij}(z')$;

(5) for $1\leqslant p<\infty$ and $\mathcal{D}$ some point set,

$$\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}):=\left\{f\colon\mathcal{D}\to\mathrm{M}_2(\mathbb{C});\ \|f(\cdot)\|_{\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})}:=\left(\int_{\mathcal{D}}|f(z)|^p\,|\mathrm{d}z|\right)^{1/p}<\infty\right\},$$

where, for $\mathcal{A}(z)\in\mathrm{M}_2(\mathbb{C})$, $|\mathcal{A}(z)|:=(\sum_{i,j=1}^2\overline{\mathcal{A}_{ij}(z)}\,\mathcal{A}_{ij}(z))^{1/2}$ is the Hilbert-Schmidt norm, with $\overline{\bullet}$ denoting complex conjugation of $\bullet$, for $p=\infty$,

$$\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}):=\left\{g\colon\mathcal{D}\to\mathrm{M}_2(\mathbb{C});\ \|g(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})}:=\max_{i,j=1,2}\ \sup_{z\in\mathcal{D}}|g_{ij}(z)|<\infty\right\},$$

and, for $f\in\mathrm{I}+\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}):=\left\{\mathrm{I}+h;\ h\in\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})\right\}$,

$$\|f(\cdot)\|_{\mathrm{I}+\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})}:=\left(\|f(\infty)\|^2_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})}+\|f(\cdot)-f(\infty)\|^2_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})}\right)^{1/2};$$

(6) for a matrix $\mathcal{A}_{ij}(z)$, $i,j=1,2$, to have boundary values in the $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})$ sense on an oriented contour $\mathcal{D}$, it is meant that $\lim_{\substack{z'\to z\\ z'\,\in\,\pm\,\text{side of}\,\mathcal{D}}}\int_{\mathcal{D}}|\mathcal{A}(z')-(\mathcal{A}(z))_\pm|^2\,|\mathrm{d}z|=0$ (e.g., if $\mathcal{D}=\mathbb{R}$ is oriented from $+\infty$ to $-\infty$, then $\mathcal{A}(z)$ has $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})$ boundary values on $\mathcal{D}$ means that $\lim_{\varepsilon\downarrow0}\int_{\mathbb{R}}|\mathcal{A}(x\mp\mathrm{i}\varepsilon)-(\mathcal{A}(x))_\pm|^2\,\mathrm{d}x=0$);

(7) for a $2\times2$ matrix-valued function $\mathfrak{Y}(z)$, the notation $\mathfrak{Y}(z)=_{z\to z_0}\mathcal{O}(\ast)$ means $\mathfrak{Y}_{ij}(z)=_{z\to z_0}\mathcal{O}(\ast_{ij})$, $i,j=1,2$ (*mutatis mutandis* for $o(1)$);

(8) $\|\mathfrak{F}(\cdot)\|_{\cap_{p\in J}\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\ast)}:=\sum_{p\in J}\|\mathfrak{F}(\cdot)\|_{\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\ast)}$, with $\mathrm{card}(J)<\infty$;

(9) $\mathcal{M}_1(\mathbb{R})$ denotes the set of all non-negative, bounded, unit Borel measures on $\mathbb{R}$ for which all moments exist,

$$\mathcal{M}_1(\mathbb{R}):=\left\{\mu;\ \int_{\mathbb{R}}\mathrm{d}\mu(s)=1,\ \int_{\mathbb{R}}s^m\mathrm{d}\mu(s)<\infty,\ m\in\mathbb{Z}\setminus\{0\}\right\};$$

(10) for $(\mu,\nu)\in\mathbb{R}\times\mathbb{R}$, denote the function $(\bullet-\mu)^{\mathrm{i}\nu}\colon\mathbb{C}\setminus(-\infty,\mu)\to\mathbb{C}$, $\bullet\mapsto\exp(\mathrm{i}\nu\ln(\bullet-\mu))$, where ln denotes the principal branch of the logarithm;

(11) for $\widetilde{\gamma}$ a nullhomologous path in a region $\mathbb{D}\subset\mathbb{C}$, $\mathrm{int}(\widetilde{\gamma}):=\left\{\zeta\in\mathbb{D}\setminus\widetilde{\gamma};\ \mathrm{ind}_{\widetilde{\gamma}}(\zeta):=\int_{\widetilde{\gamma}}\frac{1}{z-\zeta}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\neq0\right\}$;

(12) for some point set $\mathcal{D}\subset\mathcal{X}$, with $\mathcal{X}=\mathbb{C}$ or $\mathbb{R}$, $\overline{\mathcal{D}}:=\mathcal{D}\cup\partial\mathcal{D}$, and $\mathcal{D}^c:=\mathcal{X}\setminus\overline{\mathcal{D}}$.

## 2.1 Riemann Surfaces: Preliminaries

In this subsection, the basic elements associated with the construction of hyperelliptic and finite genus (compact) Riemann surfaces are presented (for further details and proofs, see, for example, [77,78]).

**Remark 2.1.1.** The superscripts $^{\pm}$, and sometimes the subscripts $_{\pm}$, in this subsection should not be confused with the subscripts $_{\pm}$ appearing in the various RHPs (this is a general comment which applies, unless stated otherwise, throughout the entire text). Although $\overline{\mathbb{C}}$ (or $\mathbb{CP}^1$) $:= \mathbb{C} \cup \{\infty\}$ (resp., $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$) is the standard definition for the (closed) Riemann sphere (resp., closed real line), the simplified, and somewhat abusive, notation $\mathbb{C}$ (resp., $\mathbb{R}$) is used to denote both the (closed) Riemann sphere, $\overline{\mathbb{C}}$ (resp., closed real line, $\overline{\mathbb{R}}$), and the (open) complex field, $\mathbb{C}$ (resp., open real line, $\mathbb{R}$), and the context(s) should make clear which object(s) the notation $\mathbb{C}$ (resp., $\mathbb{R}$) represents. ■

Let $N \in \mathbb{N}$ (with $N < \infty$ assumed throughout) and $\varsigma_k \in \mathbb{R} \setminus \{0, \pm\infty\}$, $k = 1, \dots, 2N+2$, be such that $\varsigma_i \neq \varsigma_j$ $\forall$ $i \neq j = 1, \dots, 2N+2$, and enumerated/ordered according to $\varsigma_1 < \varsigma_2 < \cdots < \varsigma_{2N+2}$. Let $R(z) := \prod_{j=1}^{N}(z-\varsigma_{2j-1})(z-\varsigma_{2j}) \in \mathbb{R}[z]$ (the algebra of polynomials in $z$ with coefficients in $\mathbb{R}$) be the (unital) polynomial of even degree $\deg(R) = 2N+2$ ($\deg(R) = 0 \pmod 2$) whose (simple) zeros/roots are $\{\varsigma_j\}_{j=1}^{2N+2}$. Denote by $\mathcal{R}$ the hyperelliptic Riemann surface of genus $N$ defined by the equation $y^2 = R(z)$ and realised as a two-sheeted branched (ramified) covering of the Riemann sphere such that its two sheets are two identical copies of $\mathbb{C}$ with branch cuts along the intervals $(\varsigma_1, \varsigma_2), (\varsigma_3, \varsigma_4), \dots, (\varsigma_{2N+1}, \varsigma_{2N+2})$, and glued/pasted to each other 'crosswise' along the opposite banks of the corresponding cuts $(\varsigma_{2j-1}, \varsigma_{2j})$, $j = 1, \dots, N+1$. Denote the two sheets of $\mathcal{R}$ by $\mathcal{R}^+$ (the first/upper sheet) and $\mathcal{R}^-$ (the second/lower sheet): to indicate that $z$ lies on the first (resp., second) sheet, one writes $z^+$ (resp., $z^-$); of course, as points in the plane $\mathbb{C}$, $z^+ = z^- = z$. For points $z$ on the first (resp., second) sheet $\mathcal{R}^+$ (resp., $\mathcal{R}^-$), one has that $z^+ = (z, +(R(z))^{1/2})$ (resp., $z^- = (z, -(R(z))^{1/2})$), where the single-valued branch of the square root is chosen such that $z^{-(N+1)}(R(z))^{1/2} \sim_{\substack{z \to \infty \\ z \in \mathcal{R}^{\pm}}} \pm 1$.

Let $\mathcal{E}_j := (\varsigma_{2j-1}, \varsigma_{2j})$, $j = 1, \dots, N+1$, and set $\mathcal{E} = \cup_{j=1}^{N+1} \mathcal{E}_j$ (note that $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$, $i \neq j = 1, \dots, N+1$). Denote by $\mathcal{E}_j^+$ ($\subset \mathcal{R}^+$) (resp., $\mathcal{E}_j^-$ ($\subset \mathcal{R}^-$)) the upper (resp., lower) bank of the interval $\mathcal{E}_j$, $j = 1, \dots, N+1$, forming $\mathcal{E}$, and oriented in accordance with the orientation of $\mathcal{E}$ as the boundary of $\mathbb{C} \setminus \mathcal{E}$, namely, the domain $\mathbb{C} \setminus \mathcal{E}$ is on the left as one proceeds along the upper bank of the $j$th interval from $\varsigma_{2j-1}$ to the point $\varsigma_{2j}$ and back along the lower bank from $\varsigma_{2j}$ to $\varsigma_{2j-1}$; thus, $\mathcal{E}_j^{\pm} := (\varsigma_{2j-1}, \varsigma_{2j})^{\pm}$, $j = 1, \dots, N+1$, are two (identical) copies of $(\varsigma_{2j-1}, \varsigma_{2j}) \subset \mathbb{R}$ 'lifted' to $\mathcal{R}^{\pm}$. Set $\Gamma := \cup_{j=1}^{N+1} \Gamma_j$ ($\subset \mathcal{R}$), where $\Gamma_j := \mathcal{E}_j^+ \cup \mathcal{E}_j^-$, $j = 1, \dots, N+1$ ($\Gamma = \mathcal{E}^+ \cup \mathcal{E}^-$): note that $\Gamma$, as a curve on $\mathcal{R}$ (defined by the equation $y^2 = R(z)$), consists of a finitely denumerable number of disjoint analytic closed Jordan curves, $\Gamma_j$, $j = 1, \dots, N+1$, which are *cycles* on $\mathcal{R}$, and that correspond to the intervals $\mathcal{E}_j$. From the above construction, it is clear that $\mathcal{R} = \mathcal{R}^+ \cup \mathcal{R}^- \cup \Gamma$; furthermore, the canonical projection of $\Gamma$ onto $\mathbb{C}$ ($\boldsymbol{\pi} \colon \mathcal{R} \to \mathbb{C}$) is $\mathcal{E}$, that is, $\boldsymbol{\pi}(\Gamma) = \mathcal{E}$ (also, $\boldsymbol{\pi}(\mathcal{R}^+) = \boldsymbol{\pi}(\mathcal{R}^-) = \mathbb{C} \setminus \mathcal{E}$, or, alternately, $\boldsymbol{\pi}(z^+) = \boldsymbol{\pi}(z^-) = z$). One moves in the 'positive (+)' (resp., 'negative (−)') direction along the (closed) contour $\Gamma \subset \mathcal{R}$ if the domain $\mathcal{R}^+$ is on the left (resp., right) and the domain $\mathcal{R}^-$ is on the right (resp., left): the corresponding notation is (see above) $\Gamma^+$ (resp., $\Gamma^-$). For a function $f$ defined on the two-sheeted hyperelliptic Riemann surface $\mathcal{R}$, one defines the non-tangential boundary values, provided they exist, of $f(z)$ as $z \in \mathcal{R}^+$ (resp., $z \in \mathcal{R}^-$) approaches $\lambda \in \Gamma$, denoted $\lambda_+$ (resp., $\lambda_-$), by $f(\lambda_{\pm}) := f_{\pm}(\lambda) := \lim_{\substack{z \to \lambda \\ z \in \Gamma^{\pm}}} f(z)$.

One takes the first $N$ contours among the (closed) contours $\Gamma_j$ for basis $\boldsymbol{\alpha}$-cycles $\{\boldsymbol{\alpha}_j, \ j = 1, \dots, N\}$ and then completes/supplements this in the standard way with $\boldsymbol{\beta}$-cycles $\{\boldsymbol{\beta}_j, \ j = 1, \dots, N\}$ so that the *intersection matrix* has the (canonical) form $\boldsymbol{\alpha}_k \circ \boldsymbol{\alpha}_j = \boldsymbol{\beta}_k \circ \boldsymbol{\beta}_j = 0$ $\forall$ $k \neq j = 1, \dots, N$, and $\boldsymbol{\alpha}_k \circ \boldsymbol{\beta}_j = \delta_{kj}$: the cycles $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}$, $j = 1, \dots, N$, form the *canonical 1-homology basis* on $\mathcal{R}$, namely, any cycle $\widehat{\boldsymbol{\gamma}} \subset \mathcal{R}$ is homologous to an integral linear combination of $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}$, that is, $\widehat{\boldsymbol{\gamma}} = \sum_{j=1}^{N}(n_j \boldsymbol{\alpha}_j + m_j \boldsymbol{\beta}_j)$, where $(n_j, m_j) \in \mathbb{Z} \times \mathbb{Z}$, $j = 1, \dots, N$. The $\boldsymbol{\alpha}$-cycles $\{\boldsymbol{\alpha}_j, \ j = 1, \dots, N\}$, in the present case, are the intervals $(\varsigma_{2j-1}, \varsigma_{2j})$, $j = 1, \dots, N$, 'going twice', that is, along the upper (from $\varsigma_{2j-1}$ to $\varsigma_{2j}$) and lower (from $\varsigma_{2j}$ to $\varsigma_{2j-1}$) banks ($\alpha_j = \mathcal{E}_j^+ \cup \mathcal{E}_j^-$, $j = 1, \dots, N$), and the $\boldsymbol{\beta}$-cycles $\{\boldsymbol{\beta}_j, \ j = 1, \dots, N\}$ are as follows: the $j$th $\boldsymbol{\beta}$-cycle consists of the $\boldsymbol{\alpha}$-cycles $\boldsymbol{\alpha}_k$, $k = j+1, \dots, N$, and the cycles 'linked' with them and consisting of (the gaps) $(\varsigma_{2k}, \varsigma_{2k+1})$, $k = 1, \dots, N$, 'going twice', that is, from $\varsigma_{2k}$ to $\varsigma_{2k+1}$ on the first sheet and in the reverse direction on the second sheet. For an arbitrary holomorphic Abelian differential (one-form) $\boldsymbol{\omega}$ on $\mathcal{R}$, the function $\int^z \boldsymbol{\omega}$ is defined uniquely modulo its $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-periods, $\oint_{\boldsymbol{\alpha}_j} \boldsymbol{\omega}$ and $\oint_{\boldsymbol{\beta}_j} \boldsymbol{\omega}$, $j = 1, \dots, N$, respectively. It is well known

that the canonical 1-homology basis $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}$, $j=1,\dots,N$, constructed above 'generates', on $\mathcal{R}$, the corresponding $\boldsymbol{\alpha}$-normalised basis of holomorphic Abelian differentials (one-forms) $\{\omega_1, \omega_2, \dots, \omega_N\}$, where $\omega_j := \sum_{k=1}^{N} \frac{c_{jk} z^{N-k}}{\sqrt{R(z)}} \, \mathrm{d}z$, $c_{jk} \in \mathbb{C}$, $j=1,\dots,N$, and $\oint_{\boldsymbol{\alpha}_k} \omega_j = \delta_{kj}$, $k,\ j=1,\dots,N$: the associated $N \times N$ matrix of $\boldsymbol{\beta}$-periods, $\tau = (\tau_{ij})_{i,j=1,\dots,N} := \left(\oint_{\boldsymbol{\beta}_j} \omega_i\right)_{i,j=1,\dots,N}$, is a *Riemann matrix*, that is, it is symmetric ($\tau_{ij} = \tau_{ji}$), pure imaginary, and $-\mathrm{i}\tau$ is positive definite ($\operatorname{Im}(\tau_{ij}) > 0$); moreover, $\tau$ is non-degenerate ($\det(\tau) \neq 0$). From the condition that the basis of the differentials $\omega_l$, $l=1,\dots,N$, is canonical, with respect to the given basis cycles $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}$, it is seen that this implies that each $\omega_l$ is real valued on $\mathcal{E} = \cup_{j=1}^{N+1} (\varsigma_{2j-1}, \varsigma_{2j})$ and has exactly one (real) root/zero in any interval (band) $(\varsigma_{2j-1}, \varsigma_{2j})$, $j=1,\dots,N+1$, $j \neq l$; moreover, in the 'gaps' $(\varsigma_{2j}, \varsigma_{2j+1})$, $j=1,\dots,N$, these differentials take non-zero, purely imaginary values.

Fix the 'standard basis' $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_N$ in $\mathbb{R}^N$, that is, $(\boldsymbol{e}_j)_k = \delta_{jk}$, $j,k=1,\dots,N$ (these standard basis vectors should be viewed as column vectors): the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_N, \tau\boldsymbol{e}_1, \tau\boldsymbol{e}_2, \dots, \tau\boldsymbol{e}_N$ are linearly independent over the real field $\mathbb{R}$, and form a 'basis' in $\mathbb{C}^N$. The quotient space $\mathbb{C}^N / \{N + \tau M\}$, $(\mathrm{N}, \mathrm{M}) \in \mathbb{Z}^N \times \mathbb{Z}^N$, where $\mathbb{Z}^N := \{(m_1, m_2, \dots, m_N);\ m_j \in \mathbb{Z},\ j=1,\dots,N\}$, is a $2N$-dimensional real torus $\mathbb{T}^{2N}$, and is referred to as the *Jacobi variety*, symbolically $\operatorname{Jac}(\mathcal{R})$, of the two-sheeted (hyperelliptic) Riemann surface $\mathcal{R}$ of genus $N$. Let $z_0$ be a fixed point in $\mathcal{R}$. A vector-valued function $\mathcal{A}(z) = (\mathcal{A}_1(z), \mathcal{A}_2(z), \dots, \mathcal{A}_N(z)) \in \operatorname{Jac}(\mathcal{R})$ with co-ordinates $\mathcal{A}_k(z) \equiv \int_{z_0}^{z} \omega_k$, $k=1,\dots,N$, where, hereafter, unless stated otherwise and/or where confusion may arise, $\equiv$ denotes 'congruence modulo the period lattice', defines the *Abel map* $\mathcal{A} \colon \mathcal{R} \to \operatorname{Jac}(\mathcal{R})$. The unordered set of points $z_1, z_2, \dots, z_N$, with $z_k \in \mathcal{R}$, form the $N$th symmetric power of $\mathcal{R}$, symbolically $\mathcal{R}^N_{\text{symm}}$ (or $\mathcal{S}^N\mathcal{R}$). The vector function $\mathfrak{U} = (\mathfrak{U}_1, \mathfrak{U}_2, \dots, \mathfrak{U}_N)$ with co-ordinates $\mathfrak{U}_j = \mathfrak{U}_j(z_1, z_2, \dots, z_N) \equiv \sum_{k=1}^{N} \mathcal{A}_j(z_k) \equiv \sum_{k=1}^{N} \int_{z_0}^{z_k} \omega_j$, $j=1,\dots,N$, that is, $(z_1, z_2, \dots, z_N) \to (\sum_{k=1}^{N} \int_{z_0}^{z_k} \omega_1, \sum_{k=1}^{N} \int_{z_0}^{z_k} \omega_2, \dots, \sum_{k=1}^{N} \int_{z_0}^{z_k} \omega_N)$, is also referred to as the *Abel map*, $\mathfrak{U} \colon \mathcal{R}^N_{\text{symm}} \to \operatorname{Jac}(\mathcal{R})$ (or $\mathfrak{U} \colon \mathcal{S}^N\mathcal{R} \to \operatorname{Jac}(\mathcal{R})$). It is known that the Abel map $\mathfrak{U} \colon \mathcal{R}^N_{\text{symm}} \to \operatorname{Jac}(\mathcal{R})$ is surjective and locally biholomorphic, but not injective globally. The *dissected* Riemann surface, symbolically $\widetilde{\mathcal{R}}$, is obtained from $\mathcal{R}$ by 'cutting' (canonical dissection) along the cycles of the canonical 1-homology basis $\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k$, $k=1,\dots,N$, of the original surface, namely, $\widetilde{\mathcal{R}} = \mathcal{R} \setminus (\cup_{j=1}^{N} (\boldsymbol{\alpha}_j \cup \boldsymbol{\beta}_j))$; the surface $\widetilde{\mathcal{R}}$ is not only connected, as one can 'pass' from one sheet to the other 'across' $\Gamma_{N+1}$, but also simply connected (a $4N$-sided polygon ($4N$-gon) of a canonical dissection of $\mathcal{R}$ associated with the given canonical 1-homology basis for $\mathcal{R}$). For a given vector $\vec{\boldsymbol{v}} = (v_1, v_2, \dots, v_N) \in \operatorname{Jac}(\mathcal{R})$, the problem of finding an unordered collection of points $z_1, z_2, \dots, z_N$, $z_j \in \mathcal{R}$, $j=1,\dots,N$, for which $\mathfrak{U}_k(z_1, z_2, \dots, z_N) \equiv v_k$, $k=1,\dots,N$, is called the *Jacobi inversion problem* for Abelian integrals: as is well known, the Jacobi inversion problem is always solvable; but not, in general, uniquely.

By a *divisor* on the Riemann surface $\mathcal{R}$ is meant a formal 'symbol' $\boldsymbol{d} = z_1^{n_f(z_1)} z_2^{n_f(z_2)} \cdots z_m^{n_f(z_m)}$, where $z_j \in \mathcal{R}$ and $n_f(z_j) \in \mathbb{Z}$, $j=1,\dots,m$: the number $|\boldsymbol{d}| := \sum_{j=1}^{m} n_f(z_j)$ is called the *degree* of the divisor $\boldsymbol{d}$: if $z_i \neq z_j$ $\forall$ $i \neq j = 1,\dots,m$, and if $n_f(z_j) \geqslant 0$, $j=1,\dots,m$, then the divisor $\boldsymbol{d}$ is said to be *integral*. Let $g$ be a meromorphic function defined on $\mathcal{R}$: for an arbitrary point $a \in \mathcal{R}$, one denotes by $n_g(a)$ (resp., $p_g(a)$) the multiplicity of the zero (resp., pole) of the function $g$ at this point if $a$ is a zero (resp., pole), and sets $n_g(a) = 0$ (resp., $p_g(a) = 0$) otherwise; thus, $n_g(a),\ p_g(a) \geqslant 0$. To a meromorphic function $g$ on $\mathcal{R}$, one assigns the divisor $(g)$ of zeros and poles of this function as $(g) = z_1^{n_g(z_1)} z_2^{n_g(z_2)} \cdots z_{l_1}^{n_g(z_{l_1})} \lambda_1^{-p_g(\lambda_1)} \lambda_2^{-p_g(\lambda_2)} \cdots \lambda_{l_2}^{-p_g(\lambda_{l_2})}$, where $z_i,\ \lambda_j \in \mathcal{R}$, $i=1,\dots,l_1$, $j=1,\dots,l_2$, are the zeros and poles of $g$ on $\mathcal{R}$, and $n_g(z_i),\ p_g(\lambda_j) \geqslant 0$ are their multiplicities (one can also write $\{(a, n_g(a), -p_g(a));\ a \in \mathcal{R},\ n_g(a),\ p_g(a) \geqslant 0\}$ for the divisor $(g)$ of $g$): these divisors are said to be *principal*.

Associated with the Riemann matrix of $\boldsymbol{\beta}$-periods, $\tau$, is the *Riemann theta function*, defined by

$$\boldsymbol{\theta}(z;\tau) =: \boldsymbol{\theta}(z) = \sum_{m \in \mathbb{Z}^N} \mathrm{e}^{2\pi\mathrm{i}(m,z) + \pi\mathrm{i}(m,\tau m)}, \quad z \in \mathbb{C}^N,$$

where $(\cdot,\cdot)$ denotes the—real—Euclidean inner/scalar product (for $\mathbf{A} = (A_1, A_2, \dots, A_N) \in \mathbb{E}^N$ and $\mathbf{B} = (B_1, B_2, \dots, B_N) \in \mathbb{E}^N$, $(A,B) := \sum_{k=1}^{N} A_k B_k$), with the following evenness and (quasi-) periodicity properties,

$$\boldsymbol{\theta}(-z) = \boldsymbol{\theta}(z), \qquad \boldsymbol{\theta}(z + e_j) = \boldsymbol{\theta}(z), \qquad \text{and} \qquad \boldsymbol{\theta}(z \pm \tau_j) = \mathrm{e}^{\mp 2\pi\mathrm{i} z_j - \mathrm{i}\pi\tau_{jj}} \boldsymbol{\theta}(z),$$

where $e_j$ is the standard (basis) column vector in $\mathbb{C}^N$ with 1 in the $j$th entry and 0 elsewhere (see above), and $\tau_j := \tau e_j$ $(\in \mathbb{C}^N)$, $j=1,\dots,N$.

It turns out that, for the analysis of this work, the following multi-valued functions are essential:

- $(R_e(z))^{1/2} := (\prod_{k=0}^{N}(z-b_k^e)(z-a_{k+1}^e))^{1/2}$, where, with the identification $a_{N+1}^e \equiv a_0^e$ (as points on the complex sphere, $\overline{\mathbb{C}}$) and with the point at infinity lying on the (open) interval $(a_0^e, b_0^e)$, $-\infty < a_0^e < b_0^e < a_1^e < b_1^e < \cdots < a_N^e < b_N^e < +\infty$, $a_0^e$ $(\equiv a_{N+1}^e) \neq -\infty, 0$, and $b_N^e \neq 0, +\infty$ (see Figure 1);

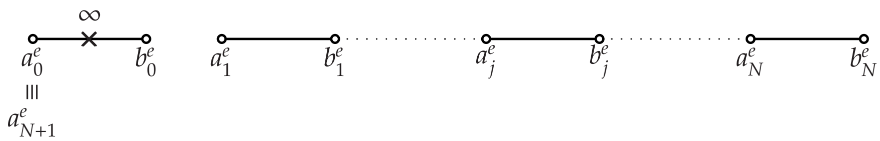


Figure 1: Union of (open) intervals in the complex $z$-plane

- $(R_o(z))^{1/2} := (\prod_{k=0}^{N}(z-b_k^o)(z-a_{k+1}^o))^{1/2}$, where, with the identification $a_{N+1}^o \equiv a_0^o$ (as points on the complex sphere, $\overline{\mathbb{C}}$) and with the point at infinity lying on the (open) interval $(a_0^o, b_0^o)$, $-\infty < a_0^o < b_0^o < a_1^o < b_1^o < \cdots < a_N^o < b_N^o < +\infty$, $a_0^o$ $(\equiv a_{N+1}^o) \neq -\infty, 0$, and $b_N^o \neq 0, +\infty$ (see Figure 2).

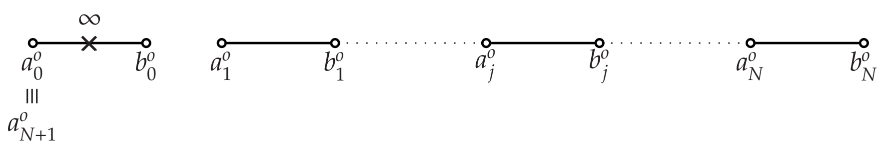


Figure 2: Union of (open) intervals in the complex $z$-plane

The functions $R_e(z)$ and $R_o(z)$, respectively, are unital polynomials ($\in \mathbb{R}[z]$) of even degree $(\deg(R_e(z)) = \deg(R_o(z)) = 2(N+1))$ whose (simple) roots/zeros are $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$ $(a_{N+1}^e \equiv a_0^e)$ and $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$ $(a_{N+1}^o \equiv a_0^o)$. The basic ingredients associated with the construction of the hyperelliptic Riemann surfaces of genus $N$ corresponding, respectively, to the multi-valued functions $y^2 = R_e(z)$ and $y^2 = R_o(z)$ was given above. One now uses the above construction; but particularised to the cases of the polynomials $R_e(z)$ and $R_o(z)$, to arrive at the following:

$\boxed{\sqrt{R_e(z)}}$ Let $\mathcal{Y}_e$ denote the two-sheeted Riemann surface of genus $N$ associated with $y^2 = R_e(z)$, with $R_e(z)$ as characterised above: the first/upper (resp., second/lower) sheet of $\mathcal{Y}_e$ is denoted by $\mathcal{Y}_e^+$ (resp., $\mathcal{Y}_e^-$), points on the first/upper (resp., second/lower) sheet are represented as $z^+ := (z, +(R_e(z))^{1/2})$ (resp., $z^- := (z, -(R_e(z))^{1/2})$), where, as points on the plane $\mathbb{C}$, $z^+ = z^- = z$, and the single-valued branch for the square root of the (multi-valued) function $(R_e(z))^{1/2}$ is chosen such that $z^{-(N+1)}(R_e(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathcal{Y}_e^{\pm}}} \pm 1$. $\mathcal{Y}_e$ is realised as a (two-sheeted) branched/ramified covering of the Riemann sphere such that its two sheets are two identical copies of $\mathbb{C}$ with branch cuts (slits) along the intervals $(a_0^e, b_0^e), (a_1^e, b_1^e), \ldots, (a_N^e, b_N^e)$ and pasted/glued together along $\cup_{j=1}^{N+1}(a_{j-1}^e, b_{j-1}^e)$ $(a_0^e \equiv a_{N+1}^e)$ in such a way that the cycles $\boldsymbol{\alpha}_0^e$ and $\{\boldsymbol{\alpha}_j^e, \boldsymbol{\beta}_j^e\}$, $j = 1, \ldots, N$, where the latter forms the canonical $\mathbf{1}$-homology basis for $\mathcal{Y}_e$, are characterised by the fact that (the closed contours) $\boldsymbol{\alpha}_j^e$, $j = 0, \ldots, N$, lie on $\mathcal{Y}_e^+$, and (the closed contours) $\boldsymbol{\beta}_j^e$, $j = 1, \ldots, N$, pass from $\mathcal{Y}_e^+$ (starting from the slit $(a_j^e, b_j^e)$), through the slit $(a_0^e, b_0^e)$ to $\mathcal{Y}_e^-$, and back again to $\mathcal{Y}_e^+$ through the slit $(a_j^e, b_j^e)$ (see Figure 3).

The canonical $\mathbf{1}$-homology basis $\{\boldsymbol{\alpha}_j^e, \boldsymbol{\beta}_j^e\}$, $j = 1, \ldots, N$, generates, on $\mathcal{Y}_e$, the (corresponding) $\boldsymbol{\alpha}^e$-normalised basis of holomorphic Abelian differentials (one-forms) $\{\omega_1^e, \omega_2^e, \ldots, \omega_N^e\}$, where $\omega_j^e := \sum_{k=1}^{N} \frac{c_{jk}^e z^{N-k}}{\sqrt{R_e(z)}}\,\mathrm{d}z$, $c_{jk}^e \in \mathbb{C}$, $j = 1, \ldots, N$, and $\oint_{\boldsymbol{\alpha}_k^e} \omega_j^e = \delta_{kj}$, $k, j = 1, \ldots, N$: $\omega_l^e$, $l = 1, \ldots, N$, is real valued on $\cup_{j=1}^{N+1}(a_{j-1}^e, b_{j-1}^e)$, and has exactly one (real) root in any (open) interval $(a_{j-1}^e, b_{j-1}^e)$, $j = 1, \ldots, N+1$; furthermore, in the intervals $(b_{j-1}^e, a_j^e)$, $j = 1, \ldots, N$, $\omega_l^e$, $l = 1, \ldots, N$, take non-zero, pure imaginary values. Let $\boldsymbol{\omega}^e := (\omega_1^e, \omega_2^e, \ldots, \omega_N^e)$ denote the basis of holomorphic one-forms on $\mathcal{Y}_e$ as normalised

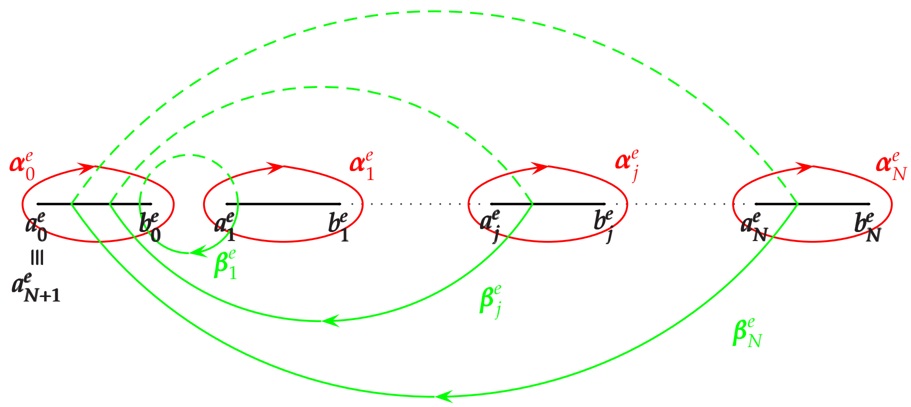


Figure 3: The Riemann surface $\mathcal{Y}_e$ of $y^2 = \prod_{k=0}^{N}(z-b_k^e)(z-a_{k+1}^e)$, $a_{N+1}^e \equiv a_0^e$. The solid (resp., dashed) lines are on the first/upper (resp., second/lower) sheet of $\mathcal{Y}_e$, denoted $\mathcal{Y}_e^+$ (resp., $\mathcal{Y}_e^-$).

above with the associated $N \times N$ Riemann matrix of $\boldsymbol{\beta}^e$-periods, $\tau^e = (\tau^e)_{i,j=1,\dots,N} := (\oint_{\boldsymbol{\beta}_j^e} \omega_i^e)_{i,j=1,\dots,N}$: the Riemann matrix, $\tau^e$, is symmetric ($\tau_{ij}^e = \tau_{ji}^e$) and pure imaginary, $-\mathrm{i}\tau^e$ is positive definite ($\mathrm{Im}(\tau_{ij}^e) > 0$), and $\det(\tau^e) \neq 0$ (non-degenerate). For the holomorphic Abelian differential (one-form) $\boldsymbol{\omega}^e$ defined above, choose $a_{N+1}^e$ as the *base point*, and set $\boldsymbol{u}^e \colon \mathcal{Y}_e \to \mathrm{Jac}(\mathcal{Y}_e)$ $(:= \mathbb{C}^N/\{N+\tau^e M\}$, $(N,M) \in \mathbb{Z}^N \times \mathbb{Z}^N)$, $z \mapsto \boldsymbol{u}^e(z) := \int_{a_{N+1}^e}^{z} \boldsymbol{\omega}^e$, where the integration from $a_{N+1}^e$ to $z$ $(\in \mathcal{Y}_e)$ is taken along any path on $\mathcal{Y}_e^+$.

**Remark 2.1.2.** From the representation $\omega_j^e = \sum_{k=1}^{N} \frac{c_{jk}^e z^{N-k}}{\sqrt{R_e(z)}} \mathrm{d}z$, $j=1,\dots,N$, and the normalisation condition $\oint_{\boldsymbol{\alpha}_k^e} \omega_j^e = \delta_{kj}$, $k,j=1,\dots,N$, one shows that $c_{jk}^e$, $k,j=1,\dots,N$, are obtained from

$$\begin{pmatrix} c_{11}^e & c_{12}^e & \cdots & c_{1N}^e \\ c_{21}^e & c_{22}^e & \cdots & c_{2N}^e \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}^e & c_{N2}^e & \cdots & c_{NN}^e \end{pmatrix} = \widetilde{\mathfrak{S}}_e^{-1}, \tag{E1}$$

where

$$\widetilde{\mathfrak{S}}_e := \begin{pmatrix} \oint_{\boldsymbol{\alpha}_1^e} \frac{\mathrm{d}s_1}{\sqrt{R_e(s_1)}} & \oint_{\boldsymbol{\alpha}_2^e} \frac{\mathrm{d}s_2}{\sqrt{R_e(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^e} \frac{\mathrm{d}s_N}{\sqrt{R_e(s_N)}} \\ \oint_{\boldsymbol{\alpha}_1^e} \frac{s_1\mathrm{d}s_1}{\sqrt{R_e(s_1)}} & \oint_{\boldsymbol{\alpha}_2^e} \frac{s_2\mathrm{d}s_2}{\sqrt{R_e(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^e} \frac{s_N\mathrm{d}s_N}{\sqrt{R_e(s_N)}} \\ \vdots & \vdots & \ddots & \vdots \\ \oint_{\boldsymbol{\alpha}_1^e} \frac{s_1^{N-1}\mathrm{d}s_1}{\sqrt{R_e(s_1)}} & \oint_{\boldsymbol{\alpha}_2^e} \frac{s_2^{N-1}\mathrm{d}s_2}{\sqrt{R_e(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^e} \frac{s_N^{N-1}\mathrm{d}s_N}{\sqrt{R_e(s_N)}} \end{pmatrix}. \tag{E2}$$

For a (representation-independent) proof of the fact that $\det(\widetilde{\mathfrak{S}}_e) \neq 0$, see, for example, Chapter 10, Section 10–2, of [77]. ∎

Set (see [38]), for $z \in \mathbb{C}_+$, $\gamma^e(z) := (\prod_{k=1}^{N+1}(z-b_{k-1}^e)(z-a_k^e)^{-1})^{1/4}$, and, for $z \in \mathbb{C}_-$, $\gamma^e(z) := -\mathrm{i}(\prod_{k=1}^{N+1}(z-b_{k-1}^e)(z-a_k^e)^{-1})^{1/4}$. It is shown in [38] that $\gamma^e(z) \underset{z \in \mathcal{Y}_e^\pm}{=}_{z\to\infty} (-\mathrm{i})^{(1\mp 1)/2}(1+O(z^{-1}))$, and

$$\left\{z_j^{e,\pm}\right\}_{j=1}^{N} = \left\{z^\pm \in \mathcal{Y}_e^\pm;\ (\gamma^e(z) \mp (\gamma^e(z))^{-1})|_{z=z^\pm} = 0\right\},$$

with $z_j^{e,\pm} \in (a_j^e, b_j^e)^\pm$ $(\subset \mathcal{Y}_e^\pm)$, $j=1,\dots,N$, where, as points on the plane, $z_j^{e,+} = z_j^{e,-} := z_j^e$, $j=1,\dots,N$ (of course, on the plane, $z_j^e \in (a_j^e, b_j^e)$, $j=1,\dots,N$).

Corresponding to $\mathcal{Y}_e$, define $\boldsymbol{d}_e := -\boldsymbol{K}_e - \sum_{j=1}^{N} \int_{a_{N+1}^e}^{z_j^{e,-}} \boldsymbol{\omega}^e$ $(\in \mathbb{C}^N)$, where $\boldsymbol{K}_e$ is the associated ('even') vector of Riemann constants, and the integration from $a_{N+1}^e$ to $z_j^{e,-}$, $j=1,\dots,N$, is taken along a

fixed path in $\mathcal{Y}_e^-$. It is shown in Chapter VII of [78] that $\boldsymbol{K}_e=\sum_{j=1}^N\int_{a_j^e}^{a_{N+1}^e}\boldsymbol{\omega}^e$; furthermore, $\boldsymbol{K}_e$ is a point of order 2, that is, $2\boldsymbol{K}_e=0$ and $s\boldsymbol{K}_e\neq 0$ for $0<s<2$. Recalling the definition of $\boldsymbol{\omega}^e$ and that $z^{-(N+1)}(R_e(z))^{1/2}\underset{z\in\mathbb{C}_\pm}{\sim_{z\to\infty}}\pm 1$, using the fact that $\boldsymbol{K}_e$ is a point of order 2, one arrives at

$$\begin{aligned}\boldsymbol{d}_e &= -\boldsymbol{K}_e-\sum_{j=1}^N\int_{a_{N+1}^e}^{z_j^{e,-}}\boldsymbol{\omega}^e=\boldsymbol{K}_e-\sum_{j=1}^N\int_{a_{N+1}^e}^{z_j^{e,-}}\boldsymbol{\omega}^e=-\boldsymbol{K}_e+\sum_{j=1}^N\int_{a_{N+1}^e}^{z_j^{e,+}}\boldsymbol{\omega}^e=\boldsymbol{K}_e+\sum_{j=1}^N\int_{a_{N+1}^e}^{z_j^{e,+}}\boldsymbol{\omega}^e\\ &=-\sum_{j=1}^N\int_{a_j^e}^{z_j^{e,-}}\boldsymbol{\omega}^e=\sum_{j=1}^N\int_{a_j^e}^{z_j^{e,+}}\boldsymbol{\omega}^e.\end{aligned}$$

Associated with the Riemann matrix of $\boldsymbol{\beta}^e$-periods, $\tau^e$, is the ('even') Riemann theta function

$$\boldsymbol{\theta}(z;\tau^e)=:\boldsymbol{\theta}^e(z)=\sum_{m\in\mathbb{Z}^N}\mathrm{e}^{2\pi\mathrm{i}(m,z)+\pi\mathrm{i}(m,\tau^e m)},\quad z\in\mathbb{C}^N;$$

$\boldsymbol{\theta}^e(z)$ has the following evenness and (quasi-) periodicity properties,

$$\boldsymbol{\theta}^e(-z)=\boldsymbol{\theta}^e(z),\qquad \boldsymbol{\theta}^e(z+e_j)=\boldsymbol{\theta}^e(z),\qquad\text{and}\qquad \boldsymbol{\theta}^e(z\pm\tau_j^e)=\mathrm{e}^{\mp 2\pi\mathrm{i}z_j-\mathrm{i}\pi\tau_{jj}^e}\boldsymbol{\theta}^e(z),$$

where $\tau_j^e:=\tau^e e_j$ ($\in\mathbb{C}^N$), $j=1,\dots,N$. This entire latter apparatus is used extensively in [38].

$\boxed{\sqrt{R_o(z)}}$ Let $\mathcal{Y}_o$ denote the two-sheeted Riemann surface of genus $N$ associated with $y^2=R_o(z)$, with $R_o(z)$ as characterised above: the first/upper (resp., second/lower) sheet of $\mathcal{Y}_o$ is denoted by $\mathcal{Y}_o^+$ (resp., $\mathcal{Y}_o^-$), points on the first/upper (resp., second/lower) sheet are represented as $z^+:=(z,+(R_o(z))^{1/2})$ (resp., $z^-:=(z,-(R_o(z))^{1/2})$), where, as points on the plane $\mathbb{C}$, $z^+=z^-=z$, and the single-valued branch for the square root of the (multi-valued) function $(R_o(z))^{1/2}$ is chosen such that $z^{-(N+1)}(R_o(z))^{1/2}\underset{z\in\mathcal{Y}_o^\pm}{\sim_{z\to\infty}}\pm 1$. $\mathcal{Y}_o$ is realised as a (two-sheeted) branched/ramified covering of the Riemann sphere such that its two sheets are two identical copies of $\mathbb{C}$ with branch cuts (slits) along the intervals $(a_0^o,b_0^o),(a_1^o,b_1^o),\dots,(a_N^o,b_N^o)$ and pasted/glued together along $\cup_{j=1}^{N+1}(a_{j-1}^o,b_{j-1}^o)$ ($a_0^o\equiv a_{N+1}^o$) in such a way that the cycles $\boldsymbol{\alpha}_0^o$ and $\{\boldsymbol{\alpha}_j^o,\boldsymbol{\beta}_j^o\}$, $j=1,\dots,N$, where the latter forms the canonical $\mathbf{1}$-homology basis for $\mathcal{Y}_o$, are characterised by the fact that (the closed contours) $\boldsymbol{\alpha}_j^o$, $j=0,\dots,N$, lie on $\mathcal{Y}_o^+$, and (the closed contours) $\boldsymbol{\beta}_j^o$, $j=1,\dots,N$, pass from $\mathcal{Y}_o^+$ (starting from the slit $(a_j^o,b_j^o)$), through the slit $(a_0^o,b_0^o)$ to $\mathcal{Y}_o^-$, and back again to $\mathcal{Y}_o^+$ through the slit $(a_j^o,b_j^o)$ (see Figure 4).

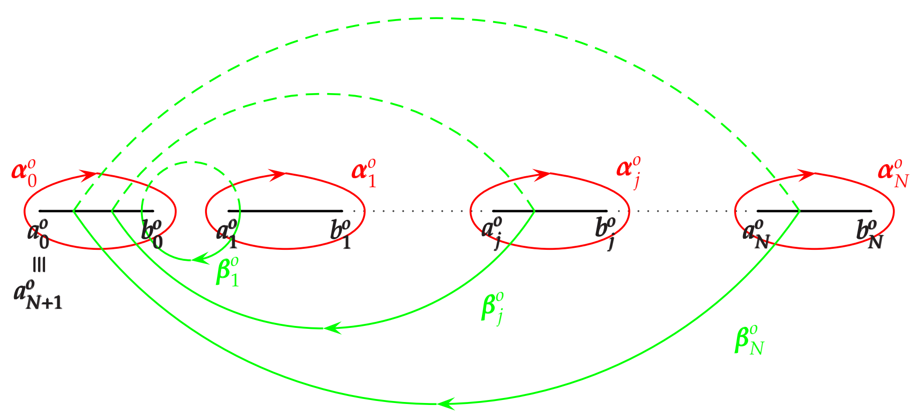


Figure 4: The Riemann surface $\mathcal{Y}_o$ of $y^2=\prod_{k=0}^{N+1}(z-b_k^o)(z-a_{k+1}^o)$, $a_{N+1}^o\equiv a_0^o$. The solid (resp., dashed) lines are on the first/upper (resp., second/lower) sheet of $\mathcal{Y}_o$, denoted $\mathcal{Y}_o^+$ (resp., $\mathcal{Y}_o^-$).

The canonical $\mathbf{1}$-homology basis $\{\boldsymbol{\alpha}_j^o, \boldsymbol{\beta}_j^o\}$, $j=1,\dots,N$, generates, on $\mathcal{Y}_o$, the (corresponding) $\boldsymbol{\alpha}^o$-normalised basis of holomorphic Abelian differentials (one-forms) $\{\omega_1^o, \omega_2^o, \dots, \omega_N^o\}$, where $\omega_j^o := \sum_{k=1}^N \frac{c_{jk}^o z^{N-k}}{\sqrt{R_o(z)}}\,\mathrm{d}z$, $c_{jk}^o \in \mathbb{C}$, $j=1,\dots,N$, and $\oint_{\boldsymbol{\alpha}_k^o} \omega_j^o = \delta_{kj}$, $k,j=1,\dots,N$: $\omega_l^o$, $l=1,\dots,N$, is real valued on $\cup_{j=1}^{N+1}(a_{j-1}^o, b_{j-1}^o)$, and has exactly one (real) root in any (open) interval $(a_{j-1}^o, b_{j-1}^o)$, $j=1,\dots,N+1$; furthermore, in the intervals $(b_{j-1}^o, a_j^o)$, $j=1,\dots,N$, $\omega_l^o$, $l=1,\dots,N$, take non-zero, pure imaginary values. Let $\boldsymbol{\omega}^o := (\omega_1^o, \omega_2^o, \dots, \omega_N^o)$ denote the basis of holomorphic one-forms on $\mathcal{Y}_o$ as normalised above with the associated $N\times N$ Riemann matrix of $\boldsymbol{\beta}^o$-periods, $\tau^o = (\tau^o)_{i,j=1,\dots,N} := (\oint_{\boldsymbol{\beta}_j^o} \omega_i^o)_{i,j=1,\dots,N}$: the Riemann matrix, $\tau^o$, is symmetric ($\tau_{ij}^o = \tau_{ji}^o$) and pure imaginary, $-\mathrm{i}\tau^o$ is positive definite ($\mathrm{Im}(\tau_{ij}^o) > 0$), and $\det(\tau^o) \neq 0$ (non-degenerate). For the holomorphic Abelian differential (one-form) $\boldsymbol{\omega}^o$ defined above, choose $a_{N+1}^o$ as the base point, and set $\boldsymbol{u}^o \colon \mathcal{Y}_o \to \mathrm{Jac}(\mathcal{Y}_o)$ $(:= \mathbb{C}^N/\{N+\tau^o M\}$, $(N,M) \in \mathbb{Z}^N \times \mathbb{Z}^N)$, $z \mapsto \boldsymbol{u}^o(z) := \int_{a_{N+1}^o}^z \boldsymbol{\omega}^o$, where the integration from $a_{N+1}^o$ to $z$ $(\in \mathcal{Y}_o)$ is taken along any path on $\mathcal{Y}_o^+$.

**Remark 2.1.3.** From the representation $\omega_j^o = \sum_{k=1}^N \frac{c_{jk}^o z^{N-k}}{\sqrt{R_o(z)}}\,\mathrm{d}z$, $j=1,\dots,N$, and the normalisation condition $\oint_{\boldsymbol{\alpha}_k^o} \omega_j^o = \delta_{kj}$, $k,j=1,\dots,N$, one shows that $c_{jk}^o$, $k,j=1,\dots,N$, are obtained from

$$\begin{pmatrix} c_{11}^o & c_{12}^o & \cdots & c_{1N}^o \\ c_{21}^o & c_{22}^o & \cdots & c_{2N}^o \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}^o & c_{N2}^o & \cdots & c_{NN}^o \end{pmatrix} = \widetilde{\mathfrak{S}}_o^{-1}, \tag{O1}$$

where

$$\widetilde{\mathfrak{S}}_o := \begin{pmatrix} \oint_{\boldsymbol{\alpha}_1^o} \frac{\mathrm{d}s_1}{\sqrt{R_o(s_1)}} & \oint_{\boldsymbol{\alpha}_2^o} \frac{\mathrm{d}s_2}{\sqrt{R_o(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^o} \frac{\mathrm{d}s_N}{\sqrt{R_o(s_N)}} \\ \oint_{\boldsymbol{\alpha}_1^o} \frac{s_1\mathrm{d}s_1}{\sqrt{R_o(s_1)}} & \oint_{\boldsymbol{\alpha}_2^o} \frac{s_2\mathrm{d}s_2}{\sqrt{R_o(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^o} \frac{s_N\mathrm{d}s_N}{\sqrt{R_o(s_N)}} \\ \vdots & \vdots & \ddots & \vdots \\ \oint_{\boldsymbol{\alpha}_1^o} \frac{s_1^{N-1}\mathrm{d}s_1}{\sqrt{R_o(s_1)}} & \oint_{\boldsymbol{\alpha}_2^o} \frac{s_2^{N-1}\mathrm{d}s_2}{\sqrt{R_o(s_2)}} & \cdots & \oint_{\boldsymbol{\alpha}_N^o} \frac{s_N^{N-1}\mathrm{d}s_N}{\sqrt{R_o(s_N)}} \end{pmatrix}. \tag{O2}$$

For a (representation-independent) proof of the fact that $\det(\widetilde{\mathfrak{S}}_o) \neq 0$, see, for example, Chapter 10, Section 10–2, of [77]. ■

Set (see Section 4), for $z \in \mathbb{C}_+$, $\gamma^o(z) := (\prod_{k=1}^{N+1}(z-b_{k-1}^o)(z-a_k^o)^{-1})^{1/4}$, and, for $z \in \mathbb{C}_-$, $\gamma^o(z) := -\mathrm{i}(\prod_{k=1}^{N+1}(z-b_{k-1}^o)(z-a_k^o)^{-1})^{1/4}$. It is shown in Section 4 that $\gamma^o(z) \underset{\substack{z\to 0\\ z\in\mathcal{Y}_o^{\pm}}}{=} (-\mathrm{i})^{(1\mp1)/2}\gamma^o(0) \cdot (1+\mathcal{O}(z))$, where $\gamma^o(0) := (\prod_{k=1}^{N+1} b_{k-1}^o (a_k^o)^{-1})^{1/4}$ $(>0)$, and a set of $N$ upper-edge and lower-edge finite-length-gap roots/zeros are

$$\left\{z_j^{o,\pm}\right\}_{j=1}^N = \left\{z^{\pm} \in \mathcal{Y}_o^{\pm};\ ((\gamma^o(0))^{-1}\gamma^o(z) \mp \gamma^o(0)(\gamma^o(z))^{-1})|_{z=z^{\pm}} = 0\right\},$$

with $z_j^{o,\pm} \in (a_j^o, b_j^o)^{\pm}$ $(\subset \mathcal{Y}_o^{\pm})$, $j=1,\dots,N$, where, as points on the plane, $z_j^{o,+} = z_j^{o,-} := z_j^o$, $j=1,\dots,N$ (of course, on the plane, $z_j^o \in (a_j^o, b_j^o)$, $j=1,\dots,N$).

Corresponding to $\mathcal{Y}_o$, define $\boldsymbol{d}_o := -\boldsymbol{K}_o - \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,-}} \boldsymbol{\omega}^o$ $(\in \mathbb{C}^N)$, where $\boldsymbol{K}_o$ is the associated ('odd') vector of Riemann constants, and the integration from $a_{N+1}^o$ to $z_j^{o,-}$, $j=1,\dots,N$, is taken along a fixed path in $\mathcal{Y}_o^-$. It is shown in Chapter VII of [78] that $\boldsymbol{K}_o = \sum_{j=1}^N \int_{a_j^o}^{a_{N+1}^o} \boldsymbol{\omega}^o$; furthermore, $\boldsymbol{K}_o$ is a point of order 2. Recalling the definition of $\boldsymbol{\omega}^o$ and that $z^{-(N+1)}(R_o(z))^{1/2} \underset{\substack{z\to\infty\\ z\in\mathbb{C}_{\pm}}}{\sim} \pm 1$, using the fact that $\boldsymbol{K}_o$ is a point of order 2, one arrives at

$$\boldsymbol{d}_o = -\boldsymbol{K}_o - \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,-}} \boldsymbol{\omega}^o = \boldsymbol{K}_o - \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,-}} \boldsymbol{\omega}^o = -\boldsymbol{K}_o + \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,+}} \boldsymbol{\omega}^o = \boldsymbol{K}_o + \sum_{j=1}^N \int_{a_{N+1}^o}^{z_j^{o,+}} \boldsymbol{\omega}^o$$

$$= -\sum_{j=1}^{N} \int_{a_j^o}^{z_j^{o,-}} \boldsymbol{\omega}^o = \sum_{j=1}^{N} \int_{a_j^o}^{z_j^{o,+}} \boldsymbol{\omega}^o.$$

Associated with the Riemann matrix of $\boldsymbol{\beta}^o$-periods, $\tau^o$, is the ('odd') Riemann theta function

$$\boldsymbol{\theta}(z;\tau^o) =: \boldsymbol{\theta}^o(z) = \sum_{m \in \mathbb{Z}^N} \mathrm{e}^{2\pi\mathrm{i}(m,z)+\pi\mathrm{i}(m,\tau^o m)}, \quad z \in \mathbb{C}^N; \tag{2.1}$$

$\boldsymbol{\theta}^o(z)$ has the following evenness and (quasi-) periodicity properties,

$$\boldsymbol{\theta}^o(-z) = \boldsymbol{\theta}^o(z), \qquad \boldsymbol{\theta}^o(z+e_j) = \boldsymbol{\theta}^o(z), \qquad \text{and} \qquad \boldsymbol{\theta}^o(z \pm \tau_j^o) = \mathrm{e}^{\mp 2\pi\mathrm{i}z_j - \mathrm{i}\pi\tau_{jj}^o}\boldsymbol{\theta}^o(z),$$

where $\tau_j^o := \tau^o e_j$ ($\in \mathbb{C}^N$), $j = 1, \ldots, N$. Extensive use of this apparatus will be made in Section 4.

## 2.2 The Riemann-Hilbert Problems for the Monic OLPs

In this subsection, the RHPs corresponding to the even degree and odd degree monic OLPs $\boldsymbol{\pi}_{2n}(z)$ and $\boldsymbol{\pi}_{2n+1}(z)$ defined, respectively, in Equations (1.4) and (1.5), are formulated *à la* Fokas-Its-Kitaev [41, 42]. Furthermore, integral representations for the even degree and odd degree monic OLPs are also obtained.

Consider the varying exponential measure $\widetilde{\mu}$ ($\in \mathcal{M}_1(\mathbb{R})$) given by $\mathrm{d}\widetilde{\mu}(z) = \mathrm{e}^{-\mathcal{N} V(z)}\,\mathrm{d}z$, $\mathcal{N} \in \mathbb{N}$, where (the external field) $V \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfies conditions (V1)–(V3). The RHPs which characterise the even degree and odd degree monic OLPs are now stated.

**RHP1.** *Let* $V \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfy conditions* (V1)–(V3). *Find* $\overset{e}{\mathrm{Y}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ *solving:* (i) $\overset{e}{\mathrm{Y}}(z)$ *is holomorphic for* $z \in \mathbb{C} \setminus \mathbb{R}$; (ii) *the boundary values* $\overset{e}{\mathrm{Y}}_\pm(z) := \lim_{\substack{z' \to z \\ \pm\mathrm{Im}(z')>0}} \overset{e}{\mathrm{Y}}(z')$ *satisfy the jump condition*

$$\overset{e}{\mathrm{Y}}_+(z) = \overset{e}{\mathrm{Y}}_-(z)\left(\mathrm{I} + \mathrm{e}^{-\mathcal{N} V(z)}\sigma_+\right), \quad z \in \mathbb{R};$$

(iii) $\overset{e}{\mathrm{Y}}(z) z^{-n\sigma_3} \underset{\substack{z \to \infty \\ z \in \mathbb{C}\setminus\mathbb{R}}}{=} \mathrm{I} + \mathcal{O}(z^{-1})$; *and* (iv) $\overset{e}{\mathrm{Y}}(z) z^{n\sigma_3} \underset{\substack{z \to 0 \\ z \in \mathbb{C}\setminus\mathbb{R}}}{=} \mathcal{O}(1)$.

**RHP2.** *Let* $V \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfy conditions* (V1)–(V3). *Find* $\overset{o}{\mathrm{Y}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ *solving:* (i) $\overset{o}{\mathrm{Y}}(z)$ *is holomorphic for* $z \in \mathbb{C} \setminus \mathbb{R}$; (ii) *the boundary values* $\overset{o}{\mathrm{Y}}_\pm(z) := \lim_{\substack{z' \to z \\ \pm\mathrm{Im}(z')>0}} \overset{o}{\mathrm{Y}}(z')$ *satisfy the jump condition*

$$\overset{o}{\mathrm{Y}}_+(z) = \overset{o}{\mathrm{Y}}_-(z)\left(\mathrm{I} + \mathrm{e}^{-\mathcal{N} V(z)}\sigma_+\right), \quad z \in \mathbb{R};$$

(iii) $\overset{o}{\mathrm{Y}}(z) z^{n\sigma_3} \underset{\substack{z \to 0 \\ z \in \mathbb{C}\setminus\mathbb{R}}}{=} \mathrm{I} + \mathcal{O}(z)$; *and* (iv) $\overset{o}{\mathrm{Y}}(z) z^{-(n+1)\sigma_3} \underset{\substack{z \to \infty \\ z \in \mathbb{C}\setminus\mathbb{R}}}{=} \mathcal{O}(1)$.

**Lemma 2.2.1.** *Let* $\overset{e}{\mathrm{Y}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ *solve* **RHP1**. **RHP1** *possesses a unique solution given by:* (i) *for* $n = 0$,

$$\overset{e}{\mathrm{Y}}(z) = \begin{pmatrix} 1 & \int_{\mathbb{R}} \frac{\exp(-\mathcal{N} V(s))}{s-z} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

*where* $\boldsymbol{\pi}_0(z) := \overset{e}{\mathrm{Y}}_{11}(z) \equiv 1$, *with* $\overset{e}{\mathrm{Y}}_{11}(z)$ *the* (1 1)*-element of* $\overset{e}{\mathrm{Y}}(z)$; *and* (ii) *for* $n \in \mathbb{N}$,

$$\overset{e}{\mathrm{Y}}(z) = \begin{pmatrix} \boldsymbol{\pi}_{2n}(z) & \int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n}(s)\exp(-\mathcal{N} V(s))}{s-z} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ \overset{e}{\mathrm{Y}}_{21}(z) & \int_{\mathbb{R}} \frac{\overset{e}{\mathrm{Y}}_{21}(s)\exp(-\mathcal{N} V(s))}{s-z} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

*where* $\overset{e}{\mathrm{Y}}_{21} \colon \mathbb{C}^* \to \mathbb{C}$ *denotes the* (2 1)*-element of* $\overset{e}{\mathrm{Y}}(z)$, *and* $\boldsymbol{\pi}_{2n}(z)$ *is the even degree monic* OLP *defined in Equation* (1.4).

*Proof.* See [38], the proof of Lemma 2.2.1. □

**Lemma 2.2.2.** *Let* $\overset{o}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to \mathrm{SL}_2(\mathbb{C})$ *solve* **RHP2**. **RHP2** *possesses a unique solution given by:* (i) *for* $n=0$,

$$\overset{o}{\mathrm{Y}}(z)=\begin{pmatrix} z\boldsymbol{\pi}_1(z) & z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_1(s))\exp(-\mathcal{N}\,V(s))}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ 2\pi\mathrm{i}z & 1+z\int_{\mathbb{R}}\frac{\exp(-\mathcal{N}\,V(s))}{s-z}\,\mathrm{d}s \end{pmatrix},\quad z\in\mathbb{C}\setminus\mathbb{R},$$

*where* $\boldsymbol{\pi}_1(z)=\frac{1}{z}+\frac{\xi_0^{(1)}}{\xi_{-1}^{(1)}}$, *with* $\frac{\xi_0^{(1)}}{\xi_{-1}^{(1)}}=-\int_{\mathbb{R}}s^{-1}\exp(-\mathcal{N}\,V(s))\,\mathrm{d}s$, $\mathcal{N}\in\mathbb{N}$; *and* (ii) *for* $n\in\mathbb{N}$,

$$\overset{o}{\mathrm{Y}}(z)=\begin{pmatrix} z\boldsymbol{\pi}_{2n+1}(z) & z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\exp(-\mathcal{N}\,V(s))}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ \overset{o}{\mathrm{Y}}_{21}(z) & z\int_{\mathbb{R}}\frac{\overset{o}{\mathrm{Y}}_{21}(s)\exp(-\mathcal{N}\,V(s))}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \end{pmatrix},\quad z\in\mathbb{C}\setminus\mathbb{R}, \tag{2.2}$$

*where* $\overset{o}{\mathrm{Y}}_{21}\colon\mathbb{C}^{*}\to\mathbb{C}$ *denotes the* $(2\,1)$*-element of* $\overset{o}{\mathrm{Y}}(z)$, *and* $\boldsymbol{\pi}_{2n+1}(z)$ *is the odd degree monic* OLP *defined in Equation* (1.5).

*Proof.* Set $\widetilde{w}(z):=\exp(-\mathcal{N}\,V(z))$, $\mathcal{N}\in\mathbb{N}$, where $V\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies conditions (V1)–(V3). Since $\int_{\mathbb{R}}s^{j}\widetilde{w}(s)\,\mathrm{d}s<\infty$, $j\in\mathbb{Z}$, and $\langle\boldsymbol{\pi}_1,\boldsymbol{\pi}_0\rangle_{\mathcal{L}}=\langle\boldsymbol{\pi}_1,1\rangle_{\mathcal{L}}=0$, it follows via an application of the Sokhotski-Plemelj formula (with the Cauchy kernel normalised at zero) that, for $n=0$, **RHP2** has the (unique) solution

$$\overset{o}{\mathrm{Y}}(z)=\begin{pmatrix} z\boldsymbol{\pi}_1(z) & z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_1(s))\widetilde{w}(s)}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ 2\pi\mathrm{i}z & 1+z\int_{\mathbb{R}}\frac{\widetilde{w}(s)}{s-z}\,\mathrm{d}s \end{pmatrix},\quad z\in\mathbb{C}\setminus\mathbb{R},$$

where $\boldsymbol{\pi}_1(z)=\frac{1}{z}+\frac{\xi_0^{(1)}}{\xi_{-1}^{(1)}}$, with $\frac{\xi_0^{(1)}}{\xi_{-1}^{(1)}}=-\int_{\mathbb{R}}s^{-1}\widetilde{w}(s)\,\mathrm{d}s$. Hereafter, $n\in\mathbb{N}$ will be considered.

If $\overset{o}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ solves **RHP2**, then it follows from the jump condition (ii) of **RHP2** that, for the elements of the first column of $\overset{o}{\mathrm{Y}}(z)$,

$$\left(\overset{o}{\mathrm{Y}}_{j1}(z)\right)_{+}=\left(\overset{o}{\mathrm{Y}}_{j1}(z)\right)_{-}:=\overset{o}{\mathrm{Y}}_{j1}(z),\quad j=1,2,$$

and, for the elements of the second row,

$$\left(\overset{o}{\mathrm{Y}}_{j2}(z)\right)_{+}-\left(\overset{o}{\mathrm{Y}}_{j2}(z)\right)_{-}=\overset{o}{\mathrm{Y}}_{j1}(z)\widetilde{w}(z),\quad j=1,2.$$

From condition (i), the normalisation condition (iii), and the boundedness condition (iv) of **RHP2**, in particular, $\overset{o}{\mathrm{Y}}_{11}(z)z^{n}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to 0}}1+\mathcal{O}(z)$, $\overset{o}{\mathrm{Y}}_{11}(z)z^{-(n+1)}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to\infty}}\mathcal{O}(1)$, $\overset{o}{\mathrm{Y}}_{21}(z)z^{n}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to 0}}\mathcal{O}(z)$, and $z^{-(n+1)}\overset{o}{\mathrm{Y}}_{21}(z)\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to\infty}}\mathcal{O}(1)$, and the fact that $\overset{o}{\mathrm{Y}}_{11}(z)$ and $\overset{o}{\mathrm{Y}}_{21}(z)$ have no jumps throughout the $z$-plane, it follows that $\overset{o}{\mathrm{Y}}_{11}(z)$ is a monic rational function with a pole at the origin and at the point at infinity, with representation $z^{-1}\overset{o}{\mathrm{Y}}_{11}(z)=\sum_{l=-n-1}^{n}\widetilde{\nu}_{l}z^{l}$, where $\widetilde{\nu}_{-n-1}=1$, and $\overset{o}{\mathrm{Y}}_{21}(z)$ is a rational function with a pole at the origin and at the point at infinity, with representation $z^{-1}\overset{o}{\mathrm{Y}}_{21}(z)=\sum_{l=-n}^{n}\nu_{l}^{\flat}z^{l}$. Application of the Sokhotski-Plemelj formula to the jump relations for $\overset{o}{\mathrm{Y}}_{j2}(z)$, $j=1,2$, gives rise to the following Cauchy-type integral representations (the Cauchy kernel is normalised at $z=0$):

$$\overset{o}{\mathrm{Y}}_{j2}(z)=z\int_{\mathbb{R}}\frac{\overset{o}{\mathrm{Y}}_{j1}(s)w(s)}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}},\quad j=1,2,\quad z\in\mathbb{C}\setminus\mathbb{R}. \tag{CA1}$$

One now studies $\overset{o}{\mathrm{Y}}_{j1}(z)$, $j=1,2$, in more detail. From the normalisation condition (iii) of **RHP2**, in particular, $\overset{o}{\mathrm{Y}}_{12}(z)z^{-n}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to 0}}\mathcal{O}(z)$ and $\overset{o}{\mathrm{Y}}_{22}(z)z^{-n}\underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to 0}}1+\mathcal{O}(z)$, the formulae (CA1), the fact that $\int_{\mathbb{R}}s^{j}\widetilde{w}(s)\,\mathrm{d}s<\infty$, $j\in\mathbb{Z}$, and the expansion (for $|z/s|\ll 1$) $\frac{1}{s-z}=\sum_{k=0}^{l}\frac{z^{k}}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^{+}$, it follows that

$$\int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{11}(s))s^{-k}\widetilde{w}(s)\,\mathrm{d}s=0,\quad k=1,2,\dots,n,\qquad\text{and}\qquad\int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{11}(s))s^{-(n+1)}\widetilde{w}(s)\,\mathrm{d}s=-2\pi\mathrm{i}\mathfrak{p}^{o},$$

for some (pure imaginary) $\mathfrak{p}^{o}$ of the form $\mathfrak{p}^{o}=\mathrm{i}\mathfrak{q}^{o}$, with $\mathfrak{q}^{o}>0$ (see below), and

$$\int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{21}(s))s^{-j}\widetilde{w}(s)\,\mathrm{d}s=0, \quad j=1,2,\dots,n-1, \qquad \text{and} \qquad \int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{21}(s))s^{-n}\widetilde{w}(s)\,\mathrm{d}s=2\pi\mathrm{i};$$

and, from the boundedness condition (iv) of **RHP2**, in particular, $\overset{o}{\mathrm{Y}}_{12}(z)z^{n+1}=_{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}\mathcal{O}(1)$ and $\overset{o}{\mathrm{Y}}_{22}(z)\cdot z^{n+1}=_{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}\mathcal{O}(1)$, the formulae (CA1), the fact that $\int_{\mathbb{R}}s^{j}\widetilde{w}(s)\,\mathrm{d}s<\infty$, $j\in\mathbb{Z}$, and the expansion (for $|s/z|\ll 1$) $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^{k}}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_{0}^{+}$, it follows that

$$\int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{11}(s))s^{k}\widetilde{w}(s)\,\mathrm{d}s=0, \quad k=0,1,\dots,n, \quad \text{and} \quad \int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{21}(s))s^{j}\widetilde{w}(s)\,\mathrm{d}s=0, \quad j=0,1,\dots,n:$$

these give rise to $2n+2$ conditions for $z^{-1}\overset{o}{\mathrm{Y}}_{11}(z)$, and $2n+1$ conditions for $z^{-1}\overset{o}{\mathrm{Y}}_{21}(z)$. Consider, first, the $2n+1$ conditions for $z^{-1}\overset{o}{\mathrm{Y}}_{21}(z)$. Recalling that the strong moments are defined by $c_{j}:=\int_{\mathbb{R}}s^{j}\widetilde{w}(s)\,\mathrm{d}s$, $j\in\mathbb{Z}$, it follows from the representation (established above) $z^{-1}\overset{o}{\mathrm{Y}}_{21}(z)=\sum_{l=-n}^{n}\nu_{l}^{\flat}z^{l}$ and the $2n+1$ conditions for $z^{-1}\overset{o}{\mathrm{Y}}_{21}(z)$ that

$$\sum_{l=-n}^{n}\nu_{l}^{\flat}c_{l+k}=0, \quad k=-(n-1),-(n-2),\dots,n, \qquad \text{and} \qquad \sum_{l=-n}^{n}\nu_{l}^{\flat}c_{l-n}=2\pi\mathrm{i},$$

that is,

$$\begin{pmatrix} c_{-2n} & c_{-2n+1} & \cdots & c_{-1} & c_{0}\\ c_{-2n+1} & c_{-2n+2} & \cdots & c_{0} & c_{1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{1} & c_{2} & \cdots & c_{2n-2} & c_{2n-1}\\ c_{0} & c_{1} & \cdots & c_{2n-1} & c_{2n} \end{pmatrix}\begin{pmatrix} \nu_{-n}^{\flat}\\ \nu_{-n+1}^{\flat}\\ \vdots\\ \nu_{n-1}^{\flat}\\ \nu_{n}^{\flat}\end{pmatrix}=\begin{pmatrix}2\pi\mathrm{i}\\ 0\\ \vdots\\ 0\\ 0\end{pmatrix}.$$

This linear system of $2n+1$ equations for the $2n+1$ unknowns $\nu_{l}^{\flat}$, $l=-n,-(n-1),\dots,n$, admits a unique solution if, and only if, the determinant of the coefficient matrix, in this case $H_{2n+1}^{(-2n)}$ (cf. Equations (1.1)), is non-zero; in fact, it will be shown that $H_{2n+1}^{(-2n)}>0$. An integral representation for the Hankel determinants $H_{k}^{(m)}$, $(m,k)\in\mathbb{Z}\times\mathbb{N}$, is now obtained; then the substitutions $m=-2n$ and $k=2n+1$ are made. In the calculations that follow, $\mathfrak{S}_{k}$ denotes the $k!$ permutations $\boldsymbol{\sigma}$ of $\{1,2,\dots,k\}$. Recalling that $c_{j}:=\int_{\mathbb{R}}s^{j}\,\mathrm{d}\widetilde{\mu}(s)$, $j\in\mathbb{Z}$, where $\mathrm{d}\widetilde{\mu}(z)=\widetilde{w}(z)\,\mathrm{d}z=\exp(-\mathcal{N}\,V(z))\,\mathrm{d}z$, and using the multi-linearity property of the determinant, via Equations (1.1), one proceeds thus (recall that $H_{0}^{(m)}:=1$):

$$\begin{aligned} H_{k}^{(m)} &:= \begin{vmatrix} c_{m} & c_{m+1} & \cdots & c_{m+k-1}\\ c_{m+1} & c_{m+2} & \cdots & c_{m+k}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+k-2} & c_{m+k-1} & \cdots & c_{m+2k-3}\\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-2}\end{vmatrix}\\ &= \begin{vmatrix} \int_{\mathbb{R}}s_{1}^{m}\,\mathrm{d}\widetilde{\mu}(s_{1}) & \int_{\mathbb{R}}s_{2}^{m+1}\,\mathrm{d}\widetilde{\mu}(s_{2}) & \cdots & \int_{\mathbb{R}}s_{k}^{m+k-1}\,\mathrm{d}\widetilde{\mu}(s_{k})\\ \int_{\mathbb{R}}s_{1}^{m+1}\,\mathrm{d}\widetilde{\mu}(s_{1}) & \int_{\mathbb{R}}s_{2}^{m+2}\,\mathrm{d}\widetilde{\mu}(s_{2}) & \cdots & \int_{\mathbb{R}}s_{k}^{m+k}\,\mathrm{d}\widetilde{\mu}(s_{k})\\ \vdots & \vdots & \ddots & \vdots\\ \int_{\mathbb{R}}s_{1}^{m+k-2}\,\mathrm{d}\widetilde{\mu}(s_{1}) & \int_{\mathbb{R}}s_{2}^{m+k-1}\,\mathrm{d}\widetilde{\mu}(s_{2}) & \cdots & \int_{\mathbb{R}}s_{k}^{m+2k-3}\,\mathrm{d}\widetilde{\mu}(s_{k})\\ \int_{\mathbb{R}}s_{1}^{m+k-1}\,\mathrm{d}\widetilde{\mu}(s_{1}) & \int_{\mathbb{R}}s_{2}^{m+k}\,\mathrm{d}\widetilde{\mu}(s_{2}) & \cdots & \int_{\mathbb{R}}s_{k}^{m+2k-2}\,\mathrm{d}\widetilde{\mu}(s_{k})\end{vmatrix}\\ &= \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k}\mathrm{d}\widetilde{\mu}(s_{1})\,\mathrm{d}\widetilde{\mu}(s_{2})\cdots\mathrm{d}\widetilde{\mu}(s_{k})\begin{vmatrix} s_{1}^{m} & s_{2}^{m+1} & \cdots & s_{k}^{m+k-1}\\ s_{1}^{m+1} & s_{2}^{m+2} & \cdots & s_{k}^{m+k}\\ \vdots & \vdots & \ddots & \vdots\\ s_{1}^{m+k-2} & s_{2}^{m+k-1} & \cdots & s_{k}^{m+2k-3}\\ s_{1}^{m+k-1} & s_{2}^{m+k} & \cdots & s_{k}^{m+2k-2}\end{vmatrix}\end{aligned}$$

$$= \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)\, s_1^m s_2^{m+1}\cdots s_k^{m+k-1} \underbrace{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_k \\ \vdots & \vdots & \ddots & \vdots \\ s_1^{k-2} & s_2^{k-2} & \cdots & s_k^{k-2} \\ s_1^{k-1} & s_2^{k-1} & \cdots & s_k^{k-1} \end{vmatrix}}_{=:\,\mathrm{V}(s_1,s_2,\ldots,s_k)}$$

$$= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_{\sigma(1)})\,\mathrm{d}\widetilde{\mu}(s_{\sigma(2)})\,\cdots\,\mathrm{d}\widetilde{\mu}(s_{\sigma(k)}) \prod_{j=1}^{k} s_{\sigma(j)}^m s_{\sigma(j)}^{j-1}\,\mathrm{V}\big(s_{\sigma(1)},s_{\sigma(2)},\ldots,s_{\sigma(k)}\big)$$

$$= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)\, s_1^m s_2^m\cdots s_k^m \left(\mathrm{sgn}(\boldsymbol{\sigma})\prod_{j=1}^{k} s_{\sigma(j)}^{j-1}\right)$$

$$\times\,\mathrm{V}(s_1,s_2,\ldots,s_k)$$

$$= \frac{1}{k!} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)\, s_1^m s_2^m\cdots s_k^m\,\mathrm{V}(s_1,s_2,\ldots,s_k)$$

$$\times \underbrace{\sum_{\sigma\in\mathfrak{S}_k}\mathrm{sgn}(\boldsymbol{\sigma}) s_{\sigma(1)}^0 s_{\sigma(2)}^1\cdots s_{\sigma(k)}^{k-1}}_{=\,\mathrm{V}(s_1,s_2,\ldots,s_k)} \quad\Rightarrow$$

$$H_k^{(m)} = \frac{1}{k!} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} \mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k)\, s_1^m s_2^m\cdots s_k^m (\mathrm{V}(s_1,s_2,\ldots,s_k))^2\,;$$

using the well-known determinantal formula $\mathrm{V}(s_1,s_2,\ldots,s_k)=\prod_{\substack{i,j=1\\ j<i}}^{k}(s_i-s_j)$, one arrives at

$$H_k^{(m)} = \frac{1}{k!} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{k} s_1^m s_2^m\cdots s_k^m \prod_{\substack{i,l=1\\ l<i}}^{k}(s_i-s_l)^2\,\mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_k), \quad (m,k)\in\mathbb{Z}\times\mathbb{N}. \tag{HA1}$$

Letting $m=-2n$ and $k=2n+1$, it follows from the formula (HA1) that

$$H_{2n+1}^{(-2n)} = \frac{1}{(2n+1)!} \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} s_1^{-2n} s_2^{-2n}\cdots s_{2n+1}^{-2n} \prod_{\substack{i,l=1\\ l<i}}^{2n+1}(s_i-s_l)^2\,\mathrm{d}\widetilde{\mu}(s_1)\,\mathrm{d}\widetilde{\mu}(s_2)\,\cdots\,\mathrm{d}\widetilde{\mu}(s_{2n+1})>0,$$

whence the existence (and uniqueness) of $z^{-1}\overset{o}{\mathrm{Y}}_{21}(z)$ (thus $\overset{o}{\mathrm{Y}}_{21}(z)$).

Similarly, it follows, from the representation (established above) $z^{-1}\overset{o}{\mathrm{Y}}_{11}(z)=\sum_{l=-n-1}^{n}\widetilde{\nu}_l z^l$, with $\widetilde{\nu}_{-n-1}=1$, and the $2n+2$ conditions for $z^{-1}\overset{o}{\mathrm{Y}}_{11}(z)$, that

$$\sum_{l=-n-1}^{n}\widetilde{\nu}_l c_{l+k}=0, \quad k=-n,-(n-1),\ldots,n, \qquad \text{and} \qquad \sum_{l=-n-1}^{n}\widetilde{\nu}_l c_{l-n-1}=-2\pi\mathrm{i}\mathfrak{p}^o,$$

that is,

$$\begin{pmatrix} 2\pi\mathrm{i} & c_{-2n-1} & \cdots & c_{-n-1} & \cdots & c_{-1} \\ 0 & c_{-2n} & \cdots & c_{-n} & \cdots & c_{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & c_{-n} & \cdots & c_{0} & \cdots & c_{n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & c_{0} & \cdots & c_{n} & \cdots & c_{2n} \end{pmatrix} \begin{pmatrix} \mathfrak{p}^{o} \\ \widetilde{\nu}_{-n} \\ \vdots \\ \widetilde{\nu}_{0} \\ \vdots \\ \widetilde{\nu}_{n} \end{pmatrix} = \begin{pmatrix} -c_{-2(n+1)} \\ -c_{-2n-1} \\ \vdots \\ -c_{-n-1} \\ \vdots \\ -c_{-1} \end{pmatrix}.$$

This linear system of $2n+2$ equations for the $2n+2$ unknowns $\widetilde{\nu}_{l}$, $l=-n,-(n-1),\cdots,n$, and $\mathfrak{p}^{o}$ admits a unique solution if, and only if, the determinant of the coefficient matrix, in this case $2\pi\mathrm{i}H^{(-2n)}_{2n+1}$, is non-zero; but, it was shown above that $H^{(-2n)}_{2n+1}>0$. Furthermore, via Cramer's Rule:

$$\mathfrak{p}^{o} = \frac{\begin{vmatrix} -c_{-2n-2} & c_{-2n-1} & \cdots & c_{-n-1} & \cdots & c_{-1} \\ -c_{-2n-1} & c_{-2n} & \cdots & c_{-n} & \cdots & c_{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -c_{-n-1} & c_{-n} & \cdots & c_{0} & \cdots & c_{n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -c_{-1} & c_{0} & \cdots & c_{n} & \cdots & c_{2n} \end{vmatrix}}{2\pi\mathrm{i}H^{(-2n)}_{2n+1}} = -\frac{1}{2\pi\mathrm{i}}\frac{H^{(-2n-2)}_{2n+2}}{H^{(-2n)}_{2n+1}}.$$

Using the Hankel determinant formula (HA1) with the substitutions $m=-2(n+1)$ and $k=2(n+1)$, one arrives at

$$H^{(-2n-2)}_{2n+2} = \frac{1}{(2n+2)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+2} s_{1}^{-2n-2}s_{2}^{-2n-2}\cdots s_{2n+2}^{-2n-2}\prod_{\substack{i,l=1\\ l<i}}^{2n+2}(s_{i}-s_{l})^{2}$$
$$\times\,\mathrm{d}\widetilde{\mu}(s_{1})\,\mathrm{d}\widetilde{\mu}(s_{2})\,\cdots\,\mathrm{d}\widetilde{\mu}(s_{2n+2})>0;$$

hence, $H^{(-2n-2)}_{2n+2}/H^{(-2n)}_{2n+1}>0$. Using, now, the fact that $\int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{11}(s))s^{k}\widetilde{w}(s)\,\mathrm{d}s=0$, $k=-n,-(n-1),\dots,n$, and the relation $\int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{11}(s))s^{-(n+1)}\widetilde{w}(s)\,\mathrm{d}s=-2\pi\mathrm{i}\mathfrak{p}^{o}$, one notes, via the above formula for $\mathfrak{p}^{o}$, that

$$\int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{11}(s))s^{-(n+1)}\widetilde{w}(s)\,\mathrm{d}s = \int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{11}(s))\underbrace{\left(s^{-(n+1)}+\widetilde{\nu}_{-n}s^{-n}+\cdots+\widetilde{\nu}_{n}s^{n}\right)}_{=s^{-1}\overset{o}{\mathrm{Y}}_{11}(s)}\widetilde{w}(s)\,\mathrm{d}s$$
$$= \int_{\mathbb{R}}(s^{-1}\overset{o}{\mathrm{Y}}_{11}(s))^{2}\widetilde{w}(s)\,\mathrm{d}s=-2\pi\mathrm{i}\mathfrak{p}^{o}=H^{(-2n-2)}_{2n+2}/H^{(-2n)}_{2n+1}\quad(>0);$$

but the right-hand side of the latter expression is equal to $(\xi^{(2n+1)}_{-n-1})^{-2}=\|{\ast}^{-1}\overset{o}{\mathrm{Y}}_{11}(\ast)\|^{2}_{\mathcal{L}}$ $(>0)$ (cf. Equations (1.8)): the existence and uniqueness of $z^{-1}\overset{o}{\mathrm{Y}}_{11}(z)=:\boldsymbol{\pi}_{2n+1}(z)$, the odd degree monic OLP with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{L}}$, is thus established. □

**Corollary 2.2.1.** *Let* $V\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (V1)–(V3). *Let* $\boldsymbol{\pi}_{2n}(z)$ *and* $\boldsymbol{\pi}_{2n+1}(z)$ *be the even degree and odd degree monic* OLPs *with respect to the inner product* $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ *defined, respectively, in Equations* (1.4) *and* (1.5), *and let* $\xi^{(2n)}_{n}$ *and* $\xi^{(2n+1)}_{-n-1}$ *be the corresponding 'even' and 'odd' norming constants, respectively. Then,* $\xi^{(2n)}_{n}$ *and* $\xi^{(2n+1)}_{-n-1}$ *have the following representations:*

$$\frac{\xi^{(2n)}_{n}}{\sqrt{2n+1}} = \sqrt{\frac{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n} s_{1}^{-2n}s_{2}^{-2n}\cdots s_{2n}^{-2n}\prod_{\substack{i,l=1\\ l<i}}^{2n}(s_{i}-s_{l})^{2}\,\mathrm{d}\widetilde{\mu}(s_{1})\,\mathrm{d}\widetilde{\mu}(s_{2})\,\cdots\,\mathrm{d}\widetilde{\mu}(s_{2n})}{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} \lambda_{1}^{-2n}\lambda_{2}^{-2n}\cdots\lambda_{2n+1}^{-2n}\prod_{\substack{i,l=1\\ l<i}}^{2n+1}(\lambda_{i}-\lambda_{l})^{2}\,\mathrm{d}\widetilde{\mu}(\lambda_{1})\,\mathrm{d}\widetilde{\mu}(\lambda_{2})\,\cdots\,\mathrm{d}\widetilde{\mu}(\lambda_{2n+1})}},$$

$$\frac{\xi^{(2n+1)}_{-n-1}}{\sqrt{2(n+1)}} = \sqrt{\frac{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} \varpi_1^{-2n}\varpi_2^{-2n}\cdots\varpi_{2n+1}^{-2n}\prod_{\substack{i,l=1\\l<i}}^{2n+1}(\varpi_i-\varpi_l)^2\,\mathrm{d}\widetilde{\mu}(\varpi_1)\,\mathrm{d}\widetilde{\mu}(\varpi_2)\cdots\mathrm{d}\widetilde{\mu}(\varpi_{2n+1})}{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+2} \varsigma_1^{-2n-2}\varsigma_2^{-2n-2}\cdots\varsigma_{2n+2}^{-2n-2}\prod_{\substack{i,l=1\\l<i}}^{2n+2}(\varsigma_i-\varsigma_l)^2\,\mathrm{d}\widetilde{\mu}(\varsigma_1)\,\mathrm{d}\widetilde{\mu}(\varsigma_2)\cdots\mathrm{d}\widetilde{\mu}(\varsigma_{2n+2})}},$$

*where* $\mathrm{d}\widetilde{\mu}(z):=\exp(-\mathcal{N}\,V(z))\,\mathrm{d}z$, $\mathcal{N}\in\mathbb{N}$.

*Proof.* Consider, without loss of generality, the representation for $\xi^{(2n+1)}_{-n-1}$. Recall that (cf. Equations (1.8)) $(\xi^{(2n+1)}_{-n-1})^2=H^{(-2n)}_{2n+1}/H^{(-2n-2)}_{2n+2}$ $(>0)$: using the integral representations for $H^{(-2n)}_{2n+1}$ and $H^{(-2n-2)}_{2n+2}$ derived in (the course of) the proof of Lemma 2.2.2, and taking positive square roots of both sides of the resulting equality, one arrives at the representation for $\xi^{(2n+1)}_{-n-1}$. See [38], Corollary 2.2.1, for the proof of the representation for $\xi^{(2n)}_{n}$. ◻

**Proposition 2.2.1.** *Let* $V\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (V1)–(V3). *Let* $\boldsymbol{\pi}_{2n}(z)$ *and* $\boldsymbol{\pi}_{2n+1}(z)$ *be the even degree and odd degree monic* OLPs *with respect to the inner product* $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ *defined, respectively, in Equations* (1.4) *and* (1.5). *Then,* $\boldsymbol{\pi}_{2n}(z)$ *and* $\boldsymbol{\pi}_{2n+1}(z)$ *have, respectively, the following integral representations:*

$$\begin{aligned}\boldsymbol{\pi}_{2n}(z) &= \frac{z^{-n}}{(2n)!H^{(-2n)}_{2n}}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n} s_0^{-2n}s_1^{-2n}\cdots s_{2n-1}^{-2n}\prod_{\substack{i,l=0\\l<i}}^{2n-1}(s_i-s_l)^2\prod_{j=0}^{2n-1}(z-s_j)\\ &\quad\times\mathrm{d}\widetilde{\mu}(s_0)\,\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n-1}),\end{aligned}$$

$$\begin{aligned}\boldsymbol{\pi}_{2n+1}(z) &= -\frac{z^{-n-1}}{(2n+1)!H^{(-2n)}_{2n+1}}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} s_0^{-2n-1}s_1^{-2n-1}\cdots s_{2n}^{-2n-1}\prod_{\substack{i,l=0\\l<i}}^{2n}(s_i-s_l)^2\prod_{j=0}^{2n}(z-s_j)\\ &\quad\times\mathrm{d}\widetilde{\mu}(s_0)\,\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n}),\end{aligned}$$

*where*

$$H^{(-2n)}_{2n}=\frac{1}{(2n)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n}\lambda_1^{-2n}\lambda_2^{-2n}\cdots\lambda_{2n}^{-2n}\prod_{\substack{i,l=1\\l<i}}^{2n}(\lambda_i-\lambda_l)^2\,\mathrm{d}\widetilde{\mu}(\lambda_1)\,\mathrm{d}\widetilde{\mu}(\lambda_2)\cdots\mathrm{d}\widetilde{\mu}(\lambda_{2n}),$$

$$H^{(-2n)}_{2n+1}=\frac{1}{(2n+1)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1}\lambda_1^{-2n}\lambda_2^{-2n}\cdots\lambda_{2n+1}^{-2n}\prod_{\substack{i,l=1\\l<i}}^{2n+1}(\lambda_i-\lambda_l)^2\,\mathrm{d}\widetilde{\mu}(\lambda_1)\,\mathrm{d}\widetilde{\mu}(\lambda_2)\cdots\mathrm{d}\widetilde{\mu}(\lambda_{2n+1}),$$

*with* $\mathrm{d}\widetilde{\mu}(z):=\exp(-\mathcal{N}\,V(z))\,\mathrm{d}z$, $\mathcal{N}\in\mathbb{N}$.

*Proof.* Consider, without loss of generality, the integral representation for the odd degree monic OLP $\boldsymbol{\pi}_{2n+1}(z)$. Let $\mathfrak{S}_k$ denote the $k!$ permutations $\boldsymbol{\sigma}$ of $\{0,1,\dots,k-1\}$. Recalling that $c_j:=\int_{\mathbb{R}}s^j\,\mathrm{d}\widetilde{\mu}(s)$, $j\in\mathbb{Z}$, where $\mathrm{d}\widetilde{\mu}(z):=\widetilde{w}(z)\,\mathrm{d}z=\exp(-\mathcal{N}\,V(z))\,\mathrm{d}z$, $\mathcal{N}\in\mathbb{N}$, with $V\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditons (V1)–(V3), and using the multi-linearity property of the determinant, via the determinantal representation for $\boldsymbol{\pi}_{2n+1}(z)$ given in Equation (1.7), one proceeds thus:

$$\boldsymbol{\pi}_{2n+1}(z) = -\frac{1}{H^{(-2n)}_{2n+1}}\begin{vmatrix} c_{-2n-1} & c_{-2n} & \cdots & c_{-1} & z^{-n-1}\\ c_{-2n} & c_{-2n+1} & \cdots & c_0 & z^{-n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{-1} & c_0 & \cdots & c_{2n-1} & z^{n-1}\\ c_0 & c_1 & \cdots & c_{2n} & z^n\end{vmatrix}$$

$$
= -\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}}\begin{vmatrix} c_{-2n-1} & c_{-2n} & \cdots & c_{-1} & c_0 \\ c_{-2n} & c_{-2n+1} & \cdots & c_0 & c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-1} & c_0 & \cdots & c_{2n-1} & c_{2n} \\ z^0 & z^1 & \cdots & z^{2n} & z^{2n+1} \end{vmatrix}
$$

$$
= -\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}}\begin{vmatrix} \int_{\mathbb{R}} s_0^{-2n-1}\,\mathrm{d}\widetilde{\mu}(s_0) & \int_{\mathbb{R}} s_0^{-2n}\,\mathrm{d}\widetilde{\mu}(s_0) & \cdots & \int_{\mathbb{R}} s_0^{0}\,\mathrm{d}\widetilde{\mu}(s_0) \\ \int_{\mathbb{R}} s_1^{-2n}\,\mathrm{d}\widetilde{\mu}(s_1) & \int_{\mathbb{R}} s_1^{-2n+1}\,\mathrm{d}\widetilde{\mu}(s_1) & \cdots & \int_{\mathbb{R}} s_1^{1}\,\mathrm{d}\widetilde{\mu}(s_1) \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathbb{R}} s_{2n}^{-1}\,\mathrm{d}\widetilde{\mu}(s_{2n}) & \int_{\mathbb{R}} s_{2n}^{0}\,\mathrm{d}\widetilde{\mu}(s_{2n}) & \cdots & \int_{\mathbb{R}} s_{2n}^{2n}\,\mathrm{d}\widetilde{\mu}(s_{2n}) \\ z^0 & z^1 & \cdots & z^{2n+1} \end{vmatrix}
$$

$$
= -\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1}\mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n})\begin{vmatrix} s_0^{-2n-1} & s_0^{-2n} & \cdots & s_0^{-1} & s_0^{0} \\ s_1^{-2n} & s_1^{-2n+1} & \cdots & s_1^{0} & s_1^{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{2n}^{-1} & s_{2n}^{0} & \cdots & s_{2n}^{2n-1} & s_{2n}^{2n} \\ z^0 & z^1 & \cdots & z^{2n} & z^{2n+1} \end{vmatrix}
$$

$$
= -\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1}\mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n})s_0^{-2n-1}s_1^{-2n}\cdots s_{2n}^{-1}
$$

$$
\times\begin{vmatrix} s_0^{0} & s_0^{1} & \cdots & s_0^{2n} & s_0^{2n+1} \\ s_1^{0} & s_1^{1} & \cdots & s_1^{2n} & s_1^{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{2n}^{0} & s_{2n}^{1} & \cdots & s_{2n}^{2n} & s_{2n}^{2n+1} \\ z^0 & z^1 & \cdots & z^{2n} & z^{2n+1} \end{vmatrix}
$$

$$
= -\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}(2n+1)!}\sum_{\boldsymbol{\sigma}\in\mathfrak{S}_{2n+1}}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1}\mathrm{d}\widetilde{\mu}(s_{\sigma(0)})\mathrm{d}\widetilde{\mu}(s_{\sigma(1)})\cdots\mathrm{d}\widetilde{\mu}(s_{\sigma(2n)})
$$

$$
\times s_{\sigma(0)}^{-2n-1}s_{\sigma(1)}^{-2n-1}\cdots s_{\sigma(2n)}^{-2n-1}s_{\sigma(0)}^{0}s_{\sigma(1)}^{1}\cdots s_{\sigma(2n)}^{2n}\begin{vmatrix} s_{\sigma(0)}^{0} & s_{\sigma(0)}^{1} & \cdots & s_{\sigma(0)}^{2n} & s_{\sigma(0)}^{2n+1} \\ s_{\sigma(1)}^{0} & s_{\sigma(1)}^{1} & \cdots & s_{\sigma(1)}^{2n} & s_{\sigma(1)}^{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{\sigma(2n)}^{0} & s_{\sigma(2n)}^{1} & \cdots & s_{\sigma(2n)}^{2n} & s_{\sigma(2n)}^{2n+1} \\ z^0 & z^1 & \cdots & z^{2n} & z^{2n+1} \end{vmatrix}
$$

$$
= -\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}(2n+1)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1}\mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n})s_0^{-2n-1}s_1^{-2n-1}\cdots s_{2n}^{-2n-1}
$$

$$
\times\left(\sum_{\boldsymbol{\sigma}\in\mathfrak{S}_{2n+1}}\mathrm{sgn}(\boldsymbol{\sigma})s_{\sigma(0)}^{0}s_{\sigma(1)}^{1}\cdots s_{\sigma(2n)}^{2n}\right)\begin{vmatrix} s_0^{0} & s_0^{1} & \cdots & s_0^{2n} & s_0^{2n+1} \\ s_1^{0} & s_1^{1} & \cdots & s_1^{2n} & s_1^{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{2n}^{0} & s_{2n}^{1} & \cdots & s_{2n}^{2n} & s_{2n}^{2n+1} \\ z^0 & z^1 & \cdots & z^{2n} & z^{2n+1} \end{vmatrix}
$$

$$
= -\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}(2n+1)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1}\mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n})s_0^{-2n-1}s_1^{-2n-1}\cdots s_{2n}^{-2n-1}
$$

$$\times\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n} \\ s_1^0 & s_1^1 & \cdots & s_1^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{2n}^0 & s_{2n}^1 & \cdots & s_{2n}^{2n} \end{vmatrix}\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n} & s_0^{2n+1} \\ s_1^0 & s_1^1 & \cdots & s_1^{2n} & s_1^{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{2n}^0 & s_{2n}^1 & \cdots & s_{2n}^{2n} & s_{2n}^{2n+1} \\ z^0 & z^1 & \cdots & z^{2n} & z^{2n+1} \end{vmatrix};$$

but a straightforward calculation shows that

$$\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n} & s_0^{2n+1} \\ s_1^0 & s_1^1 & \cdots & s_1^{2n} & s_1^{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{2n}^0 & s_{2n}^1 & \cdots & s_{2n}^{2n} & s_{2n}^{2n+1} \\ z^0 & z^1 & \cdots & z^{2n} & z^{2n+1} \end{vmatrix} = \begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n} \\ s_1^0 & s_1^1 & \cdots & s_1^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{2n}^0 & s_{2n}^1 & \cdots & s_{2n}^{2n} \end{vmatrix} \prod_{j=0}^{2n}(z-s_j),$$

whence

$$\begin{aligned}
\boldsymbol{\pi}_{2n}(z) = &-\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}(2n+1)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} \mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n})s_0^{-2n-1}s_1^{-2n-1}\cdots s_{2n}^{-2n-1} \\
&\times\prod_{j=0}^{2n}(z-s_j)\begin{vmatrix} s_0^0 & s_0^1 & \cdots & s_0^{2n} \\ s_1^0 & s_1^1 & \cdots & s_1^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{2n}^0 & s_{2n}^1 & \cdots & s_{2n}^{2n} \end{vmatrix}^2 \\
= &-\frac{z^{-n-1}}{H_{2n+1}^{(-2n)}(2n+1)!}\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}_{2n+1} \mathrm{d}\widetilde{\mu}(s_0)\mathrm{d}\widetilde{\mu}(s_1)\cdots\mathrm{d}\widetilde{\mu}(s_{2n})s_0^{-2n-1}s_1^{-2n-1}\cdots s_{2n}^{-2n-1} \\
&\times\prod_{j=0}^{2n}(z-s_j)\underbrace{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ s_0^1 & s_1^1 & \cdots & s_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ s_0^{2n} & s_1^{2n} & \cdots & s_{2n}^{2n} \end{vmatrix}^2}_{=\prod_{\substack{i,l=0\\ l<i}}^{2n}(s_i-s_l)^2};
\end{aligned}$$

hence the integral representation for $\boldsymbol{\pi}_{2n+1}(z)$ stated in the Proposition, with the integral representation for $H_{2n+1}^{(-2n)}$ derived in the proof of Lemma 2.2.2. See [38], Proposition 2.2.1, for the proof of the integral representation for the even degree monic OLP $\boldsymbol{\pi}_{2n}(z)$. □

**Remark 2.2.1.** For the purposes of the ensuing asymptotic analysis, it is convenient to re-write $\mathrm{d}\widetilde{\mu}(z)=\exp(-\mathcal{N}\,V(z))\,\mathrm{d}z=\exp(-n\widetilde{V}(z))\,\mathrm{d}z=:\mathrm{d}\mu(z)$, $n\in\mathbb{N}$, where

$$\widetilde{V}(z)=z_o V(z),$$

with

$$z_o\colon \mathbb{N}\times\mathbb{N}\to\mathbb{R}_+,\ (\mathcal{N},n)\mapsto z_o:=\mathcal{N}/n,$$

and where the 'scaled' external field $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies the following conditions:

$$\widetilde{V} \text{ is real analytic on } \mathbb{R}\setminus\{0\}; \tag{2.3}$$

$$\lim_{|x|\to\infty}\Big(\widetilde{V}(x)/\ln(x^2+1)\Big)=+\infty; \tag{2.4}$$

$$\lim_{|x|\to 0}\Big(\widetilde{V}(x)/\ln(x^{-2}+1)\Big)=+\infty. \tag{2.5}$$

(For example, a rational function of the form $\widetilde{V}(z)=\sum_{k=-2m_1}^{2m_2}\widetilde{\varrho}_k z^k$, with $\widetilde{\varrho}_k\in\mathbb{R}$, $k=-2m_1,\dots,2m_2$, $m_{1,2}\in\mathbb{N}$, and $\widetilde{\varrho}_{-2m_1},\widetilde{\varrho}_{2m_2}>0$ would satisfy conditions (2.3)–(2.5).)

Hereafter, the double-scaling limit as $\mathcal{N},n\to\infty$ such that $z_o=1+o(1)$ is studied (the simplified 'notation' $n\to\infty$ will be adopted). ■

It is, by now, a well-known, if not established, mathematical fact that variational conditions for minimisation problems in logarithmic potential theory, via the *equilibrium measure* [43, 44, 79–81], play a crucial rôle in the asymptotic analysis of (matrix) RHPs associated with (continuous and discrete) orthogonal polynomials, their roots, and corresponding recurrence relation coefficients (see, for example, [46, 47, 49, 53, 62]). The situation with respect to the large-$n$ asymptotic analysis for the monic OLPs, $\boldsymbol{\pi}_n(z)$, is analogous; but, unlike the asymptotic analysis for the orthogonal polynomials case, the asymptotic analysis for $\boldsymbol{\pi}_n(z)$ requires the consideration of two different families of RHPs, one for even degree (**RHP1**) and one for odd degree (**RHP2**). Thus, one must consider two sets of variational conditions for two (suitably posed) minimisation problems.

The following discussion is decomposed into two parts: one part corresponding to the RHP for $\overset{e}{\mathrm{Y}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ formulated as **RHP1**, denoted by $\mathbf{P_1}$, and the other part corresponding to the RHP for $\overset{o}{\mathrm{Y}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ formulated as **RHP2**, denoted by $\mathbf{P_2}$.

$\boxed{\mathbf{P_1}}$ Let $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfy conditions (2.3)–(2.5). Let $\mathrm{I}_V^e[\mu^e] \colon \mathcal{M}_1(\mathbb{R}) \to \mathbb{R}$ denote the functional

$$\mathrm{I}_V^e[\mu^e] = \iint_{\mathbb{R}^2} \ln\left(\frac{|st|}{|s-t|^2}\right) \mathrm{d}\mu^e(s)\, \mathrm{d}\mu^e(t) + 2\int_{\mathbb{R}} \widetilde{V}(s)\, \mathrm{d}\mu^e(s),$$

and consider the associated minimisation problem,

$$E_V^e = \inf\Big\{\mathrm{I}_V^e[\mu^e];\ \mu^e \in \mathcal{M}_1(\mathbb{R})\Big\}.$$

The infimum is finite, and there exists a unique measure $\mu_V^e$, referred to as the 'even' equilibrium measure, achieving the infimum (that is, $\mathcal{M}_1(\mathbb{R}) \ni \mu_V^e = \inf\{\mathrm{I}_V^e[\mu^e];\ \mu^e \in \mathcal{M}_1(\mathbb{R})\}$). Furthermore, $\mu_V^e$ has the following 'regularity' properties (see [38] for complete details and proofs):

- the 'even' equilibrium measure has compact support which consists of the disjoint union of a finite number of bounded real intervals; in fact, as shown in [38], $\mathrm{supp}(\mu_V^e) =: J_e$[3] $= \cup_{j=1}^{N+1}(b_{j-1}^e, a_j^e)$ $(\subset \mathbb{R} \setminus \{0\})$, where $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$, with $b_0^e := \min\{\mathrm{supp}(\mu_V^e)\} \notin \{-\infty, 0\}$, $a_{N+1}^e := \max\{\mathrm{supp}(\mu_V^e)\} \notin \{0, +\infty\}$, and $-\infty < b_0^e < a_1^e < b_1^e < a_2^e < \cdots < b_N^e < a_{N+1}^e < +\infty$, constitute the end-points of the support of $\mu_V^e$;
- the end-points $\{b_{j-1}^e, a_j^e\}_{j=1}^{N+1}$ are not arbitrary; rather, they satisfy an $n$-dependent and (locally) solvable system of $2(N+1)$ *moment conditions* (transcendental equations; see [38], Lemma 3.5);
- the 'even' equilibrium measure is absolutely continuous with respect to Lebesgue measure. The *density* is given by

$$\mathrm{d}\mu_V^e(x) := \psi_V^e(x)\, \mathrm{d}x = \frac{1}{2\pi\mathrm{i}}(R_e(x))_+^{1/2} h_V^e(x) \mathbf{1}_{J_e}(x)\, \mathrm{d}x,$$

where

$$(R_e(z))^{1/2} := \left(\prod_{j=1}^{N+1}(z-b_{j-1}^e)(z-a_j^e)\right)^{1/2},$$

with $(R_e(x))_\pm^{1/2} := \lim_{\varepsilon\downarrow 0}(R_e(x \pm \mathrm{i}\varepsilon))^{1/2}$ and the branch of the square root is chosen, as per the discussion in Subsection 2.1, such that $z^{-(N+1)}(R_e(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathbb{C}_\pm}} \pm 1$, $h_V^e(z) := \frac{1}{2}\oint_{C_{\mathrm{R}}^e}(R_e(s))^{-1/2}(\frac{\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{2\pi})(s-z)^{-1}\, \mathrm{d}s$ (real analytic for $z \in \mathbb{R} \setminus \{0\}$), where $'$ denotes differentiation with respect to the argument, $C_{\mathrm{R}}^e$ $(\subset \mathbb{C}^*)$ is the union of two circular contours, one outer one of large radius $R^\natural$ traversed clockwise and one inner one of small radius $r^\natural$ traversed counter-clockwise, with the numbers $0 < r^\natural < R^\natural < +\infty$ chosen such that, for (any) non-real $z$ in the domain of analyticity of $\widetilde{V}$ (that is, $\mathbb{C}^*$), $\mathrm{int}(C_{\mathrm{R}}^e) \supset J_e \cup \{z\}$, and $\mathbf{1}_{J_e}(x)$ denotes the indicator

[3] It would be more usual, from the outset, for the bounded (and closed) set $\overline{J_e} := \cup_{j=1}^{N+1}[b_{j-1}^e, a_j^e]$ to denote the support of $\mu_V^e$; however, the open (and bounded) set $J_e$ provides an effective description of (the interior of) the support of $\mu_V^e$: for this reason, $J_e$ (and at other times $\overline{J_e}$) is used to denote $\mathrm{supp}(\mu_V^e)$; *mutatis mutandis* for $J_o$ and $\overline{J_o}$ (see $\boxed{\mathrm{P}_2}$ below). This should not cause confusion for the reader.

(characteristic) function of the set $J_e$. (Note that $\psi_V^e(x) \geqslant 0 \ \forall \ x \in \overline{J_e} := \cup_{j=1}^{N+1}[b_{j-1}^e, a_j^e]$: it vanishes like a square root at the end-points of the support of the 'even' equilibrium measure, that is, $\psi_V^e(s) =_{s \downarrow b_{j-1}^e} \mathcal{O}((s-b_{j-1}^e)^{1/2})$ and $\psi_V^e(s) =_{s \uparrow a_j^e} \mathcal{O}((a_j^e - s)^{1/2})$, $j=1,\dots,N+1$.);

- the 'even' equilibrium measure and its (compact) support are uniquely characterised by the following Euler-Lagrange variational equations: there exists $\ell_e \in \mathbb{R}$, the 'even' Lagrange multiplier, and $\mu^e \in \mathcal{M}_1(\mathbb{R})$ such that

$$4 \int_{J_e} \ln(|x-s|) \, \mathrm{d}\mu^e(s) - 2 \ln |x| - \widetilde{V}(x) - \ell_e = 0, \quad x \in \overline{J_e}, \tag{$\mathrm{P}_1^{(a)}$}$$

$$4 \int_{J_e} \ln(|x-s|) \, \mathrm{d}\mu^e(s) - 2 \ln |x| - \widetilde{V}(x) - \ell_e \leqslant 0, \quad x \in \mathbb{R} \setminus \overline{J_e}; \tag{$\mathrm{P}_1^{(b)}$}$$

- the Euler-Lagrange variational equations can be conveniently recast in terms of the complex potential $g^e(z)$ of $\mu_V^e$:

$$g^e(z) := \int_{J_e} \ln\Big((z-s)^2 (zs)^{-1}\Big) \, \mathrm{d}\mu_V^e(s), \quad z \in \mathbb{C} \setminus (-\infty, \max\{0, a_{N+1}^e\}).$$

The function $g^e \colon \mathbb{C} \setminus (-\infty, \max\{0, a_{N+1}^e\}) \to \mathbb{C}$ so defined satisfies:

($\mathrm{P}_1^{(1)}$) $g^e(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, \max\{0, a_{N+1}^e\})$;

($\mathrm{P}_1^{(2)}$) $g^e(z) =_{\substack{z \to \infty \\ z \in \mathbb{C} \setminus \mathbb{R}}} \ln(z) + \mathcal{O}(1)$;

($\mathrm{P}_1^{(3)}$) $g_+^e(z) + g_-^e(z) - \widetilde{V}(z) - \ell_e + 2Q_e = 0$, $z \in \overline{J_e}$, where $g_\pm^e(z) := \lim_{\varepsilon \downarrow 0} g^e(z \pm \mathrm{i}\varepsilon)$, and $Q_e := \int_{J_e} \ln(s) \, \mathrm{d}\mu_V^e(s) = \int_{J_e} \ln(|s|) \, \mathrm{d}\mu_V^e(s) + \mathrm{i}\pi \int_{J_e \cap \mathbb{R}_-} \mathrm{d}\mu_V^e(s)$;

($\mathrm{P}_1^{(4)}$) $g_+^e(z) + g_-^e(z) - \widetilde{V}(z) - \ell_e + 2Q_e \leqslant 0$, $z \in \mathbb{R} \setminus \overline{J_e}$, where equality holds for at most a finite number of points;

($\mathrm{P}_1^{(5)}$) $g_+^e(z) - g_-^e(z) = \mathrm{i} f_{g^e}^{\mathbb{R}}(z)$, $z \in \mathbb{R}$, where $f_{g^e}^{\mathbb{R}} \colon \mathbb{R} \to \mathbb{R}$, and, in particular, $g_+^e(z) - g_-^e(z) = \mathrm{i}\,\mathrm{const.}$, $z \in \mathbb{R} \setminus \overline{J_e}$, with $\mathrm{const.} \in \mathbb{R}$;

($\mathrm{P}_1^{(6)}$) $\mathrm{i}(g_+^e(z) - g_-^e(z))' \geqslant 0$, $z \in J_e$, where equality holds for at most a finite number of points.

**$\mathrm{P}_2$** Let $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfy conditions (2.3)–(2.5). Let $\mathrm{I}_V^o[\mu^o] \colon \mathcal{M}_1(\mathbb{R}) \to \mathbb{R}$ denote the functional

$$\mathrm{I}_V^o[\mu^o] = \iint_{\mathbb{R}^2} \ln\left(\frac{|st|}{|s-t|^{2+\frac{1}{n}}}\right) \mathrm{d}\mu^o(s) \, \mathrm{d}\mu^o(t) + 2 \int_{\mathbb{R}} \widetilde{V}(s) \, \mathrm{d}\mu^o(s), \quad n \in \mathbb{N},$$

and consider the associated minimisation problem,

$$E_V^o = \inf\Big\{\mathrm{I}_V^o[\mu^o]; \ \mu^o \in \mathcal{M}_1(\mathbb{R})\Big\}.$$

The infimum is finite, and there exists a unique measure $\mu_V^o$, referred to as the 'odd' equilibrium measure, achieving the infimum (that is, $\mathcal{M}_1(\mathbb{R}) \ni \mu_V^o = \inf\{\mathrm{I}_V^o[\mu^o]; \ \mu^o \in \mathcal{M}_1(\mathbb{R})\}$). Furthermore, $\mu_V^o$ has the following 'regularity' properties (all of these properties are proven in this work):

- the 'odd' equilibrium measure has compact support which consists of the disjoint union of a finite number of bounded real intervals; in fact, as shown in Section 3 (see Lemma 3.5), $\mathrm{supp}(\mu_V^o) =: J_o = \cup_{j=1}^{N+1}(b_{j-1}^o, a_j^o)$ ($\subset \mathbb{R} \setminus \{0\}$), where $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$, with $b_0^o := \min\{\mathrm{supp}(\mu_V^o)\} \notin \{-\infty, 0\}$, $a_{N+1}^o := \max\{\mathrm{supp}(\mu_V^o)\} \notin \{0, +\infty\}$, and $-\infty < b_0^o < a_1^o < b_1^o < a_2^o < \cdots < b_N^o < a_{N+1}^o < +\infty$, constitute the end-points of the support of $\mu_V^o$; (The number of intervals, $N+1$, is the same in the 'odd' case as in the 'even' case, which can be established by a lengthy analysis similar to that contained in [81].)
- the end-points $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$ are not arbitrary; rather, they satisfy the $n$-dependent and (locally) solvable system of $2(N+1)$ moment conditions (transcendental equations) given in Lemma 3.5;
- the 'odd' equilibrium measure is absolutely continuous with respect to Lebesgue measure. The density is given by

$$\mathrm{d}\mu_V^o(x) := \psi_V^o(x) \, \mathrm{d}x = \frac{1}{2\pi\mathrm{i}} (R_o(x))_+^{1/2} h_V^o(x) \mathbf{1}_{J_o}(x) \, \mathrm{d}x,$$

where

$$(R_o(z))^{1/2}:=\left(\prod_{j=1}^{N+1}(z-b_{j-1}^o)(z-a_j^o)\right)^{1/2},$$

with $(R_o(x))_{\pm}^{1/2}:=\lim_{\varepsilon\downarrow 0}(R_o(x\pm \mathrm{i}\varepsilon))^{1/2}$ and the branch of the square root is chosen, as per the discussion in Subsection 2.1, such that $z^{-(N+1)}(R_o(z))^{1/2}\sim_{\substack{z\to\infty\\ z\in\mathbb{C}_{\pm}}}\pm 1$, $h_V^o(z):=(2+\frac{1}{n})^{-1}\oint_{C_{\mathrm{R}}^o}(R_o(s))^{-1/2}(\frac{\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{2\pi})(s-z)^{-1}\,\mathrm{d}s$ (real analytic for $z\in\mathbb{R}\setminus\{0\}$), where $C_{\mathrm{R}}^o$ ($\subset\mathbb{C}^*$) is the union of two circular contours, one outer one of large radius $R^{\flat}$ traversed clockwise and one inner one of small radius $r^{\flat}$ traversed counter-clockwise, with the numbers $0<r^{\flat}<R^{\flat}<+\infty$ chosen such that, for (any) non-real $z$ in the domain of analyticity of $\widetilde{V}$ (that is, $\mathbb{C}^*$), $\mathrm{int}(C_{\mathrm{R}}^o)\supset J_o\cup\{z\}$, and $\mathbf{1}_{J_o}(x)$ denotes the indicator (characteristic) function of the set $J_o$. (Note that $\psi_V^o(x)\geqslant 0\ \forall\ x\in\overline{J_o}:=\cup_{j=1}^{N+1}[b_{j-1}^o,a_j^o]$: it vanishes like a square root at the end-points of the support of the 'odd' equilibrium measure, that is, $\psi_V^o(s)=_{s\downarrow b_{j-1}^o}\mathcal{O}((s-b_{j-1}^o)^{1/2})$ and $\psi_V^o(s)=_{s\uparrow a_j^o}\mathcal{O}((a_j^o-s)^{1/2})$, $j=1,\dots,N+1$.);

- the 'odd' equilibrium measure and its (compact) support are uniquely characterised by the following Euler-Lagrange variational equations: there exists $\ell_o\in\mathbb{R}$, the 'odd' Lagrange multiplier, and $\mu^o\in\mathcal{M}_1(\mathbb{R})$ such that

$$2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|x-s|)\,\mathrm{d}\mu^o(s)-2\ln|x|-\widetilde{V}(x)-\ell_o-2\left(2+\frac{1}{n}\right)\widetilde{Q}_o=0,\quad x\in\overline{J_o},\qquad (\mathrm{P}_2^{(a)})$$

$$2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|x-s|)\,\mathrm{d}\mu^o(s)-2\ln|x|-\widetilde{V}(x)-\ell_o-2\left(2+\frac{1}{n}\right)\widetilde{Q}_o\leqslant 0,\quad x\in\mathbb{R}\setminus\overline{J_o},\qquad (\mathrm{P}_2^{(b)})$$

where $\widetilde{Q}_o:=\int_{J_o}\ln(|s|)\,\mathrm{d}\mu^o(s)$;

- the Euler-Lagrange variational equations can be conveniently recast in terms of the complex potential $g^o(z)$ of $\mu_V^o$:

$$g^o(z):=\int_{J_o}\ln\left((z-s)^{2+\frac{1}{n}}(zs)^{-1}\right)\mathrm{d}\mu_V^o(s),\quad z\in\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^o\}).$$

The function $g^o\colon\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^o\})\to\mathbb{C}$ so defined satisfies:
($\mathrm{P}_2^{(1)}$) $g^o(z)$ is analytic for $z\in\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^o\})$;
($\mathrm{P}_2^{(2)}$) $g^o(z)=_{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}-\ln(z)+\mathcal{O}(1)$;
($\mathrm{P}_2^{(3)}$) $g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+-\mathfrak{Q}_{\mathcal{A}}^-=0$, $z\in\overline{J_o}$, where $g_{\pm}^o(z):=\lim_{\varepsilon\downarrow 0}g^o(z\pm\mathrm{i}\varepsilon)$, and $\mathfrak{Q}_{\mathcal{A}}^{\pm}:=(1+\frac{1}{n})\int_{J_o}\ln(|s|)\,\mathrm{d}\mu_V^o(s)-\mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\mathrm{d}\mu_V^o(s)\pm\mathrm{i}\pi(2+\frac{1}{n})\int_{J_o\cap\mathbb{R}_+}\mathrm{d}\mu_V^o(s)$;
($\mathrm{P}_2^{(4)}$) $g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+-\mathfrak{Q}_{\mathcal{A}}^-\leqslant 0$, $z\in\mathbb{R}\setminus\overline{J_o}$, where equality holds for at most a finite number of points;
($\mathrm{P}_2^{(5)}$) $g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^++\mathfrak{Q}_{\mathcal{A}}^-=\mathrm{i}f_{g^o}^{\mathbb{R}}(z)$, $z\in\mathbb{R}$, where $f_{g^o}^{\mathbb{R}}\colon\mathbb{R}\to\mathbb{R}$, and, in particular, $g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^++\mathfrak{Q}_{\mathcal{A}}^-=\mathrm{i}\,\mathrm{const.}$, $z\in\mathbb{R}\setminus\overline{J_o}$, with $\mathrm{const.}\in\mathbb{R}$;
($\mathrm{P}_2^{(6)}$) $\mathrm{i}(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^++\mathfrak{Q}_{\mathcal{A}}^-)'\geqslant 0$, $z\in J_o$, where equality holds for at most a finite number of points.

In this three-fold series of works on asymptotics of OLPs and related quantities, the so-called 'regular case' is studied, namely:

- $\mathrm{d}\mu_V^e$, or $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5), is *regular* if: (i) $h_V^e(x)\not\equiv 0$ on $\overline{J_e}$; (ii) $4\int_{J_e}\ln(|x-s|)\,\mathrm{d}\mu_V^e(s)-2\ln|x|-\widetilde{V}(x)-\ell_e<0$, $x\in\mathbb{R}\setminus\overline{J_e}$; and (iii) inequalities ($\mathrm{P}_1^{(4)}$) and ($\mathrm{P}_1^{(6)}$) in $\mathbf{P_1}$ are strict, that is, $\leqslant$ (resp., $\geqslant$) is replaced by $<$ (resp., $>$);
- $\mathrm{d}\mu_V^o$, or $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5), is regular if: (i) $h_V^o(x)\not\equiv 0$ on $\overline{J_o}$; (ii) $2(2+\frac{1}{n})\int_{J_o}\ln(|x-s|)\,\mathrm{d}\mu_V^o(s)-2\ln|x|-\widetilde{V}(x)-\ell_o-2(2+\frac{1}{n})Q_o<0$, $x\in\mathbb{R}\setminus\overline{J_o}$, where $Q_o:=\int_{J_o}\ln(|s|)\,\mathrm{d}\mu_V^o(s)$; and (iii) inequalities ($\mathrm{P}_2^{(4)}$) and ($\mathrm{P}_2^{(6)}$) in $\mathbf{P_2}$ are strict, that is, $\leqslant$ (resp., $\geqslant$) is replaced by $<$ (resp., $>$)[4].

[4]There are three distinct situations in which these conditions may fail: (i) for at least one $\widetilde{x}_e\in\mathbb{R}\setminus\overline{J_e}$ (resp., $\widetilde{x}_o\in\mathbb{R}\setminus\overline{J_o}$),

The (density of the) 'even' and 'odd' equilibrium measures $\mathrm{d}\mu_V^e$ and $\mathrm{d}\mu_V^o$, respectively, together with the corresponding variational problems, emerge naturally in the asymptotic analyses of **RHP1** and **RHP2**.

**Remark 2.2.2.** The following correspondences should also be noted:

- $g^e \colon \mathbb{C} \setminus (-\infty, \max\{0, a_{N+1}^e\}) \to \mathbb{C}$ solves the *phase conditions* $(\mathrm{P}_1^{(1)})$–$(\mathrm{P}_1^{(6)}) \Leftrightarrow \mathcal{M}_1(\mathbb{R}) \ni \mu_V^e$ solves the variational conditions $(\mathrm{P}_1^{(a)})$ and $(\mathrm{P}_1^{(b)})$;
- $g^o \colon \mathbb{C} \setminus (-\infty, \max\{0, a_{N+1}^o\}) \to \mathbb{C}$ solves the phase conditions $(\mathrm{P}_2^{(1)})$–$(\mathrm{P}_2^{(6)}) \Leftrightarrow \mathcal{M}_1(\mathbb{R}) \ni \mu_V^o$ solves the variational conditions $(\mathrm{P}_2^{(a)})$ and $(\mathrm{P}_2^{(b)})$. ■

Since the main results of this paper are asymptotics (as $n \to \infty$) for $\boldsymbol{\pi}_{2n+1}(z)$ $(z \in \mathbb{C})$, $\xi_{-n-1}^{(2n+1)}$ and $\phi_{2n+1}(z)$ $(z \in \mathbb{C})$, which are, via Lemma 2.2.2, Equation (2.2), and Equations (1.3) and (1.5), related to **RHP2** for $\overset{o}{\mathrm{Y}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$, no further reference, henceforth, to **RHP1** (and Lemma 2.2.1) for $\overset{e}{\mathrm{Y}} \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ will be made (see [38] for the complete details of the asymptotic analysis of **RHP1**). In the ensuing analysis, the large-$n$ behaviour of the solution of **RHP2** (see Lemma 2.2.2, Equation (2.2)), hence asymptotics for $\boldsymbol{\pi}_{2n+1}(z)$ (in the entire complex plane), $\xi_{-n-1}^{(2n+1)}$ and $\phi_{2n+1}(z)$ (in the entire complex plane), are extracted.

## 2.3 Summary of Results

In this subsection, the final results of this work are presented (see Sections 3–5 for the detailed analyses and proofs). Before doing so, however, some notational preamble is necessary. For $j = 1, \dots, N+1$, let

$$\Phi_{a_j}^o(z) := \left(\frac{3}{2}\left(n+\frac{1}{2}\right)\int_{a_j^o}^{z} (R_o(s))^{1/2} h_V^o(s)\,\mathrm{d}s\right)^{2/3},$$

and

$$\Phi_{b_{j-1}}^o(z) := \left(-\frac{3}{2}\left(n+\frac{1}{2}\right)\int_{z}^{b_{j-1}^o} (R_o(s))^{1/2} h_V^o(s)\,\mathrm{d}s\right)^{2/3},$$

where $(R_o(z))^{1/2}$ and $h_V^o(z)$ are defined in Theorem 2.3.1, Equations (2.8) and (2.9). Define the 'small', mutually disjoint open discs about the end-points of the support of the 'odd' equilibrium measure, $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$, as follows: for $j = 1, \dots, N+1$,

$$\mathbb{U}_{\delta_{a_j}}^o := \left\{z \in \mathbb{C};\ |z - a_j^o| < \delta_{a_j}^o\right\} \qquad \text{and} \qquad \mathbb{U}_{\delta_{b_{j-1}}}^o := \left\{z \in \mathbb{C};\ |z - b_{j-1}^o| < \delta_{b_{j-1}}^o\right\},$$

where $(0,1) \ni \delta_{a_j}^o$ (resp., $(0,1) \ni \delta_{b_{j-1}}^o$) are chosen 'sufficiently small' so that $\Phi_{a_j}^o(z)$ (resp., $\Phi_{b_{j-1}}^o(z)$), which are bi-holomorphic, conformal, and orientation preserving (resp., bi-holomorphic, conformal, and non-orientation preserving), map $\mathbb{U}_{\delta_{a_j}}^o$ (resp., $\mathbb{U}_{\delta_{b_{j-1}}}^o$), as well as the oriented skeletons (see Figure 5) $\cup_{l=1}^4 \Sigma_{a_j}^{o,l}$ (resp., $\cup_{l=1}^4 \Sigma_{b_{j-1}}^{o,l}$ (see Figure 6)), injectively onto open (and convex), $n$-dependent neighbourhoods of 0 such that:

**(i)** $\Phi_{a_j}^o(a_j^o) = 0$ (resp., $\Phi_{b_{j-1}}^o(b_{j-1}^o) = 0$);

**(ii)** $\Phi_{a_j}^o \colon \mathbb{U}_{\delta_{a_j}}^o \to \widehat{\mathbb{U}}_{\delta_{a_j}}^o := \Phi_{a_j}^o(\mathbb{U}_{\delta_{a_j}}^o)$ (resp., $\Phi_{b_{j-1}}^o \colon \mathbb{U}_{\delta_{b_{j-1}}}^o \to \widehat{\mathbb{U}}_{\delta_{b_{j-1}}}^o := \Phi_{b_{j-1}}^o(\mathbb{U}_{\delta_{b_{j-1}}}^o)$);

4 $\int_{J_e} \ln(|\widetilde{x}_e - s|)\,\mathrm{d}\mu_V^e(s) - 2\ln|\widetilde{x}_e| - \widetilde{V}(\widetilde{x}_e) - \ell_e = 0$ (resp., $2(2+\frac{1}{n})\int_{J_o} \ln(|\widetilde{x}_o - s|)\,\mathrm{d}\mu_V^o(s) - 2\ln|\widetilde{x}_o| - \widetilde{V}(\widetilde{x}_o) - \ell_o - 2(2+\frac{1}{n})Q_o = 0$), that is, for $n$ even (resp., $n$ odd) equality is attained for at least one point $\widetilde{x}_e$ (resp., $\widetilde{x}_o$) in the complement of the closure of the support of the 'even' (resp., 'odd') equilibrium measure $\mu_V^e$ (resp., $\mu_V^o$), which corresponds to the situation in which a 'band' has just closed, or is about to open, about $\widetilde{x}_e$ (resp., $\widetilde{x}_o$); (ii) for at least one $\widehat{x}_e$ (resp., $\widehat{x}_o$), $h_V^e(\widehat{x}_e) = 0$ (resp., $h_V^o(\widehat{x}_o) = 0$), that is, for $n$ even (resp., $n$ odd) the function $h_V^e$ (resp., $h_V^o$) vanishes for at least one point $\widehat{x}_e$ (resp., $\widehat{x}_o$) within the support of the 'even' (resp., 'odd') equilibrium measure $\mu_V^e$ (resp., $\mu_V^o$), which corresponds to the situation in which a 'gap' is about to open, or close, about $\widehat{x}_e$ (resp., $\widehat{x}_o$); and (iii) there exists at least one $j \in \{1, \dots, N+1\}$, denoted $j_e$ (resp., $j_o$), such that $h_V^e(b_{j_e-1}^e) = 0$ and/or $h_V^e(a_{j_e}^e) = 0$ (resp., $h_V^o(b_{j_o-1}^o) = 0$ and/or $h_V^o(a_{j_o}^o) = 0$). Each of these three cases can occur only a finite number of times due to the fact that $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfies conditions (2.3)–(2.5) [46, 81].

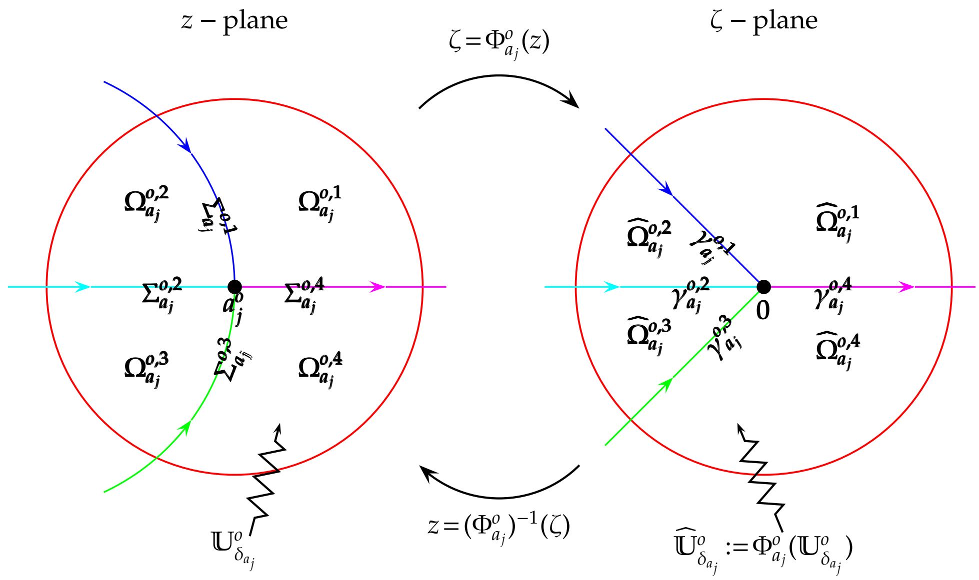


Figure 5: The conformal mapping $\zeta = \Phi^{o}_{a_j}(z) := (\frac{3}{2}(n+\frac{1}{2}) \int_{a^{o}_{j}}^{z} (R_o(s))^{1/2} h^{o}_{V}(s))^{2/3}$, $j=1,\dots,N+1$, where $(\Phi^{o}_{a_j})^{-1}$ denotes the inverse mapping

**(iii)** $\Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}} \cap \Sigma^{o,l}_{a_j}) = \Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}}) \cap \gamma^{o,l}_{a_j}$ (resp., $\Phi^{o}_{b_{j-1}}(\mathbb{U}^{o}_{\delta_{b_{j-1}}} \cap \Sigma^{o,l}_{b_{j-1}}) = \Phi^{o}_{b_{j-1}}(\mathbb{U}^{o}_{\delta_{b_{j-1}}}) \cap \gamma^{o,l}_{b_{j-1}}$);

**(iv)** $\Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}} \cap \Omega^{o,l}_{a_j}) = \Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}}) \cap \widehat{\Omega}^{o,l}_{a_j}$ (resp., $\Phi^{o}_{b_{j-1}}(\mathbb{U}^{o}_{\delta_{b_{j-1}}} \cap \Omega^{o,l}_{b_{j-1}}) = \Phi^{o}_{b_{j-1}}(\mathbb{U}^{o}_{\delta_{b_{j-1}}}) \cap \widehat{\Omega}^{o,l}_{b_{j-1}}$), with $\widehat{\Omega}^{o,1}_{a_j}$ (and $\widehat{\Omega}^{o,1}_{b_{j-1}}$) $= \{\zeta \in \mathbb{C};\, \arg(\zeta) \in (0, 2\pi/3)\}$, $\widehat{\Omega}^{o,2}_{a_j}$ (and $\widehat{\Omega}^{o,2}_{b_{j-1}}$) $= \{\zeta \in \mathbb{C};\, \arg(\zeta) \in (2\pi/3, \pi)\}$, $\widehat{\Omega}^{o,3}_{a_j}$ (and $\widehat{\Omega}^{o,3}_{b_{j-1}}$) $= \{\zeta \in \mathbb{C};\, \arg(\zeta) \in (-\pi, -2\pi/3)\}$, and $\widehat{\Omega}^{o,4}_{a_j}$ (and $\widehat{\Omega}^{o,4}_{b_{j-1}}$) $= \{\zeta \in \mathbb{C};\, \arg(\zeta) \in (-2\pi/3, 0)\}$[5].

Introduce, now, the Airy function, $\mathrm{Ai}(\cdot)$, which appears in several of the final results of this work: $\mathrm{Ai}(\cdot)$ is determined (uniquely) as the solution of the second-order, non-constant coefficient, homogeneous ODE (see, for example, Chapter 10 of [82])

$$\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0,$$

with asymptotics (at infinity)

$$\begin{aligned} \mathrm{Ai}(z) &\underset{\substack{z\to\infty \\ |\arg z|<\pi}}{\sim} \frac{1}{2\sqrt{\pi}} z^{-1/4}\,\mathrm{e}^{-\widehat{\zeta}(z)} \sum_{k=0}^{\infty} (-1)^k s_k (\widehat{\zeta}(z))^{-k}, \qquad \widehat{\zeta}(z) := \frac{2}{3} z^{3/2}, \\ \mathrm{Ai}'(z) &\underset{\substack{z\to\infty \\ |\arg z|<\pi}}{\sim} -\frac{1}{2\sqrt{\pi}} z^{1/4}\,\mathrm{e}^{-\widehat{\zeta}(z)} \sum_{k=0}^{\infty} (-1)^k t_k (\widehat{\zeta}(z))^{-k}, \end{aligned} \tag{2.6}$$

where $s_0 = t_0 = 1$,

$$s_k = \frac{\Gamma(3k+1/2)}{54^k k!\,\Gamma(k+1/2)} = \frac{(2k+1)(2k+3)\cdots(6k-1)}{216^k k!}, \qquad t_k = -\left(\frac{6k+1}{6k-1}\right) s_k, \quad k \in \mathbb{N},$$

and $\Gamma(\cdot)$ is the gamma (factorial) function.

In order to present the final asymptotic (as $n \to \infty$) results, and for arbitrary $j = 1, \dots, N+1$, consider the following decomposition (see Figure 7), into bounded and unbounded regions, of $\mathbb{C}$ and the neighbourhoods of the end-points $b^{o}_{i-1}, a^{o}_{i}$, $i = 1, \dots, N+1$ (as per the discussion above, $\mathbb{U}^{o}_{\delta_{b_{k-1}}} \cap \mathbb{U}^{o}_{\delta_{a_k}} = \varnothing$,

[5] The precise angles between the sectors are not absolutely important; one could, for example, replace $2\pi/3$ by any angle strictly between 0 and $\pi$ [2, 46, 47, 49, 79].

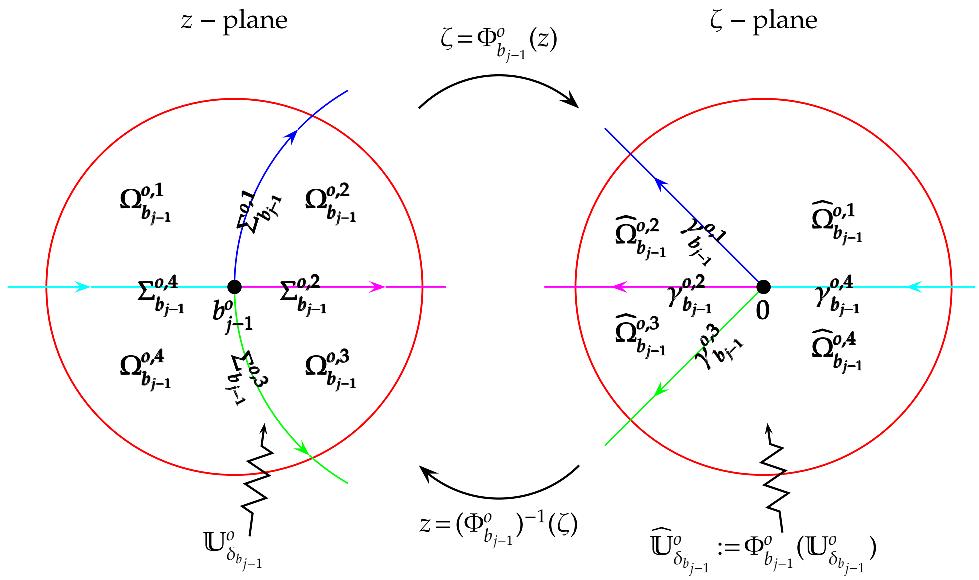


Figure 6: The conformal mapping $\zeta = \Phi^{o}_{b_{j-1}}(z) := (-\frac{3}{2}(n+\frac{1}{2})\int_{z}^{b^{o}_{j-1}}(R_o(s))^{1/2}h^{o}_{V}(s))^{2/3}$, $j=1,\dots,N+1$, where $(\Phi^{o}_{b_{j-1}})^{-1}$ denotes the inverse mapping

$k=1,\dots,N+1$). Asymptotics (as $n \to \infty$) for $\boldsymbol{\pi}_{2n+1}(z)$, with $z \in \cup_{j=1}^{4}(\Upsilon^{o}_{j} \cup (\cup_{k=1}^{N+1}(\Omega^{o,j}_{b_{k-1}} \cup \Omega^{o,j}_{a_k})))$, are now presented. These asymptotic expansions are obtained via a union of the DZ non-linear steepest-descent method [1,2] and the extension of Deift-Venakides-Zhou [3] (see, also, [45–63,66–69], and the pedagogical exposition [79]).

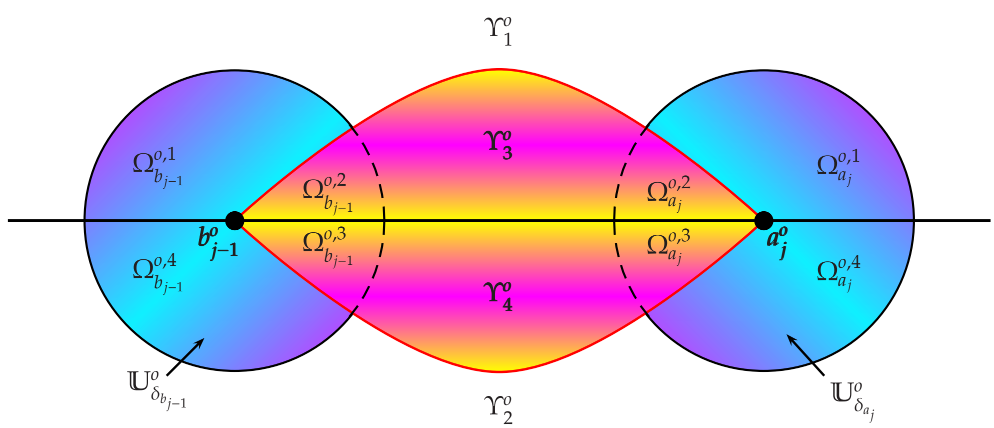


Figure 7: Region-by-region decomposition of $\mathbb{C}$ and the neighbourhoods surrounding the end-points of the support of the 'odd' equilibrium measure, $\{b^{o}_{j-1}, a^{o}_{j}\}_{j=1}^{N+1}$

**Remark 2.3.1.** In order to eschew a flood of superfluous notation, the simplified 'notation' $\mathcal{O}((n+1/2)^{-2})$ is maintained throughout Theorem 2.3.1 (see below), and is to be understood in the following, *normal* sense: for a compact subset, $\mathfrak{D}$, say, of $\mathbb{C}$, and uniformly with respect to $z \in \mathfrak{D}$, $\mathcal{O}((n+1/2)^{-2}) := \mathcal{O}(c^{\natural}(z,n)(n+1/2)^{-2})$, where $\|c^{\natural}(\cdot,n)\|_{\mathcal{L}^{p}(\mathfrak{D})} =_{n\to\infty} \mathcal{O}(1)$, $p \in \{1,2,\infty\}$, and $\exists\ K_{\mathfrak{D}} > 0$ (and finite) such that, $\forall\ z \in \mathfrak{D}$, $|c^{\natural}(z,n)| \leqslant_{n\to\infty} K_{\mathfrak{D}}$. ■

**Theorem 2.3.1.** *Let the external field* $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5). *Set*

$$\mathrm{d}\mu_V^o(x):=\psi_V^o(x)\,\mathrm{d}x=\frac{1}{2\pi\mathrm{i}}(R_o(x))_+^{1/2}h_V^o(x)\mathbf{1}_{J_o}(x)\,\mathrm{d}x, \tag{2.7}$$

*where*

$$(R_o(z))^{1/2}:=\left(\prod_{k=1}^{N+1}(z-b_{k-1}^o)(z-a_k^o)\right)^{1/2}, \tag{2.8}$$

*with* $(R_o(x))_\pm^{1/2}:=\lim_{\varepsilon\downarrow 0}(R_o(x\pm\mathrm{i}\varepsilon))^{1/2}$, $x\in J_o:=\operatorname{supp}(\mu_V^o)=\cup_{j=1}^{N+1}(b_{j-1}^o,a_j^o)$ $(\subset\mathbb{R}\setminus\{0\})$, $N\in\mathbb{N}$ *(and finite),* $b_0^o:=\min\{\operatorname{supp}(\mu_V^o)\}\notin\{-\infty,0\}$, $a_{N+1}^o:=\max\{\operatorname{supp}(\mu_V^o)\}\notin\{0,+\infty\}$, *and* $-\infty<b_0^o<a_1^o<b_1^o<a_2^o<\cdots<b_N^o<a_{N+1}^o<+\infty$, *and the branch of the square root is chosen so that* $z^{-(N+1)}(R_o(z))^{1/2}\sim_{\substack{z\to\infty\\ z\in\mathbb{C}_\pm}}\pm 1$,

$$h_V^o(z):=\frac{1}{2}\left(2+\frac{1}{n}\right)^{-1}\oint_{C_{\mathrm{R}}^o}\frac{\left(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi}\right)}{\sqrt{R_o(s)}\,(s-z)}\,\mathrm{d}s \tag{2.9}$$

*(real analytic for* $z\in\mathbb{R}\setminus\{0\}$*),* $C_{\mathrm{R}}^o$ $(\subset\mathbb{C}^*)$ *is the boundary of any open doubly-connected annular region of the type* $\{z'\in\mathbb{C};\ 0<r^\flat<|z'|<R^\flat<+\infty\}$, *where the simple outer (resp., inner) boundary* $\{z'=R^\flat\mathrm{e}^{\mathrm{i}\vartheta},\ 0\leqslant\vartheta\leqslant 2\pi\}$ *(resp.,* $\{z'=r^\flat\mathrm{e}^{\mathrm{i}\vartheta},\ 0\leqslant\vartheta\leqslant 2\pi\}$*) is traversed clockwise (resp., counter-clockwise), with the numbers* $0<r^\flat<R^\flat<+\infty$ *chosen such that, for (any) non-real* $z$ *in the domain of analyticity of* $\widetilde{V}$ *(that is,* $\mathbb{C}^*$*),* $\operatorname{int}(C_{\mathrm{R}}^o)\supset J_o\cup\{z\}$, $\mathbf{1}_{J_o}(x)$ *denotes the indicator (characteristic) function of the set* $J_o$, *and* $\{b_{j-1}^o,a_j^o\}_{j=1}^{N+1}$ *satisfy the following* $n$*-dependent and (locally) solvable system of* $2(N+1)$ *moment conditions:*

$$\begin{gathered}\int_{J_o}\frac{(2s^{-1}+\widetilde{V}'(s))s^j}{(R_o(s))_+^{1/2}}\,\mathrm{d}s=0,\quad j=0,\dots,N,\qquad \int_{J_o}\frac{(2s^{-1}+\widetilde{V}'(s))s^{N+1}}{(R_o(s))_+^{1/2}}\frac{\mathrm{d}s}{2\pi\mathrm{i}}=-\left(2+\frac{1}{n}\right),\\ \int_{a_j^o}^{b_j^o}\left(\frac{\mathrm{i}(R_o(s))^{1/2}}{2\pi}\int_{J_o}\frac{(2\xi^{-1}+\widetilde{V}'(\xi))}{(R_o(\xi))_+^{1/2}(\xi-s)}\,\mathrm{d}\xi\right)\mathrm{d}s=\ln\left|\frac{a_j^o}{b_j^o}\right|+\frac{1}{2}\left(\widetilde{V}(a_j^o)-\widetilde{V}(b_j^o)\right),\quad j=1,\dots,N.\end{gathered} \tag{2.10}$$

*Suppose, furthermore, that* $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *is regular, namely:*

(i) $h_V^o(x)\not\equiv 0$ *on* $\overline{J_o}:=J_o\cup\left(\cup_{k=1}^{N+1}\{b_{k-1}^o,a_k^o\}\right)$;

(ii)

$$2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|x-s|)\,\mathrm{d}\mu_V^o(s)-2\ln|x|-\widetilde{V}(x)-\ell_o-2\left(2+\frac{1}{n}\right)Q_o=0,\quad x\in\overline{J_o}, \tag{2.11}$$

*which defines the 'odd' variational constant* $\ell_o\in\mathbb{R}$ *(the same on each—compact—interval* $[b_{j-1}^o,a_j^o]$, $j=1,\dots,N+1$*), where*

$$Q_o:=\int_{J_o}\ln(|s|)\,\mathrm{d}\mu_V^o(s), \tag{2.12}$$

*and*

$$2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|x-s|)\,\mathrm{d}\mu_V^o(s)-2\ln|x|-\widetilde{V}(x)-\ell_o-2\left(2+\frac{1}{n}\right)Q_o<0,\quad x\in\mathbb{R}\setminus\overline{J_o};$$

(iii)

$$g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\left(\mathfrak{Q}_{\mathcal{A}}^++\mathfrak{Q}_{\mathcal{A}}^-\right)<0,\quad z\in\mathbb{R}\setminus\overline{J_o},$$

*where*

$$g^o(z):=\int_{J_o}\ln\left((z-s)^{2+\frac{1}{n}}(zs)^{-1}\right)\mathrm{d}\mu_V^o(s),\quad z\in\mathbb{C}\setminus(-\infty,\max\{0,a_{N+1}^o\}), \tag{2.13}$$

*and*

$$\mathfrak{Q}_{\mathcal{A}}^\pm:=\left(1+\frac{1}{n}\right)\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s-\mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\psi_V^o(s)\,\mathrm{d}s\pm\mathrm{i}\pi\left(2+\frac{1}{n}\right)\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s,$$

*with*

$$\int_{J_o\cap\mathbb{R}_-}\psi_V^o(s)\,\mathrm{d}s=\begin{cases}0, & J_o\subset\mathbb{R}_+,\\ 1, & J_o\subset\mathbb{R}_-,\\ \int_{b_0^o}^{a_j^o}\psi_V^o(s)\,\mathrm{d}s, & (a_j^o,b_j^o)\ni 0, \quad j=1,\dots,N,\end{cases}$$

$$\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s=\begin{cases}0, & J_o\subset\mathbb{R}_-,\\ 1, & J_o\subset\mathbb{R}_+,\\ \int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s, & (a_j^o,b_j^o)\ni 0, \quad j=1,\dots,N;\end{cases}$$

(iv)

$$\mathrm{i}\Big(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^-\Big)'>0, \quad z\in J_o.$$

*Set*

$$\overset{o}{m}{}^{\infty}(z)=\begin{cases}\overset{o}{\mathfrak{M}}{}^{\infty}(z), & z\in\mathbb{C}_+,\\ -\mathrm{i}\,\overset{o}{\mathfrak{M}}{}^{\infty}(z)\sigma_2, & z\in\mathbb{C}_-,\end{cases} \tag{2.14}$$

*where* $(\det(\overset{o}{m}{}^{\infty}(z))=1)$

$$\overset{o}{\mathfrak{M}}{}^{\infty}(z)=\begin{pmatrix}\frac{((\gamma^o(0))^{-1}\gamma^o(z)+\gamma^o(0)(\gamma^o(z))^{-1})}{2}\mathfrak{m}_{11}^o(z) & -\frac{((\gamma^o(0))^{-1}\gamma^o(z)-\gamma^o(0)(\gamma^o(z))^{-1})}{2\mathrm{i}}\mathfrak{m}_{12}^o(z)\\ \frac{((\gamma^o(0))^{-1}\gamma^o(z)-\gamma^o(0)(\gamma^o(z))^{-1})}{2\mathrm{i}}\mathfrak{m}_{21}^o(z) & \frac{((\gamma^o(0))^{-1}\gamma^o(z)+\gamma^o(0)(\gamma^o(z))^{-1})}{2}\mathfrak{m}_{22}^o(z)\end{pmatrix}, \tag{2.15}$$

$$\gamma^o(z):=\left(\left(\frac{z-b_0^o}{z-a_{N+1}^o}\right)\prod_{k=1}^{N}\left(\frac{z-b_k^o}{z-a_k^o}\right)\right)^{1/4}, \qquad \gamma^o(0):=\left(\prod_{k=1}^{N+1}\frac{b_{k-1}^o}{a_k^o}\right)^{1/4} \quad (>0), \tag{2.16}$$

$$\mathfrak{m}_{11}^o(z):=\frac{1}{\mathbb{E}}\frac{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)+\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}^o(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}^o(z)+\boldsymbol{d}_o)}, \tag{2.17}$$

$$\mathfrak{m}_{12}^o(z):=\frac{1}{\mathbb{E}}\frac{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)+\boldsymbol{d}_o)\boldsymbol{\theta}^o(-\boldsymbol{u}^o(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)\boldsymbol{\theta}^o(-\boldsymbol{u}^o(z)+\boldsymbol{d}_o)}, \tag{2.18}$$

$$\mathfrak{m}_{21}^o(z):=\mathbb{E}\frac{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)+\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}^o(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(-\boldsymbol{u}_+^o(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}^o(z)-\boldsymbol{d}_o)}, \tag{2.19}$$

$$\mathfrak{m}_{22}^o(z):=\mathbb{E}\frac{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)+\boldsymbol{d}_o)\boldsymbol{\theta}^o(-\boldsymbol{u}^o(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(-\boldsymbol{u}_+^o(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}^o(z)+\boldsymbol{d}_o)}, \tag{2.20}$$

*with*

$$\mathbb{E}=\exp\left(\mathrm{i}2\pi\left(n+\frac{1}{2}\right)\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s\right),$$

$$\boldsymbol{u}^o(z)=\int_{a_{N+1}^o}^{z}\boldsymbol{\omega}^o, \qquad \boldsymbol{u}_+^o(0)=\int_{a_{N+1}^o}^{0^+}\boldsymbol{\omega}^o,$$

$\boldsymbol{\Omega}^o=(\Omega_1^o,\Omega_2^o,\dots,\Omega_N^o)^{\mathrm{T}}$ $(\in\mathbb{R}^N)$, *where*

$$\Omega_j^o:=4\pi\int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s, \quad j=1,\dots,N,$$

*and*

$$\boldsymbol{d}_o\equiv\sum_{j=1}^{N}\int_{a_j^o}^{z_j^{o,+}}\boldsymbol{\omega}^o \quad \left(=-\sum_{j=1}^{N+1}\int_{a_j^o}^{z_j^{o,-}}\boldsymbol{\omega}^o\right),$$

*where a set of N upper-edge and lower-edge finite-length-gap roots/zeros are*

$$\left\{z_j^{o,\pm}\right\}_{j=1}^{N}=\left\{z^{\pm}\in\mathbb{C}_{\pm};\ ((\gamma^o(0))^{-1}\gamma^o(z)\mp\gamma^o(0)(\gamma^o(z))^{-1})|_{z=z^{\pm}}=0\right\},$$

*with* $z_j^{o,\pm} \in (a_j^o, b_j^o)^{\pm}$ $(\subset \mathbb{C}_{\pm})$, $j=1,\dots,N$.

*Let* $\overset{o}{\mathrm{Y}} \colon \mathbb{C}\setminus\mathbb{R} \to \mathrm{SL}_2(\mathbb{C})$ *be the unique solution of* **RHP2** *whose integral representations are given in Lemma* 2.2.2; *in particular,* $z\boldsymbol{\pi}_{2n+1}(z) := (\overset{o}{\mathrm{Y}}(z))_{11}$. *Then:*

**(1)** *for* $z \in \Upsilon_1^o$ $(\subset \mathbb{C}_+)$,

$$
\begin{aligned}
z\boldsymbol{\pi}_{2n+1}(z) \underset{n\to\infty}{=} \mathbb{E}\exp\Big(n(g^o(z)-\mathfrak{Q}_{\mathcal{A}}^+)\Big)\Bigg( & (\overset{o}{m}{}^{\infty}(z))_{11}\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{11} \\
& + \mathcal{O}\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg) + (\overset{o}{m}{}^{\infty}(z))_{21}\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{12} + \mathcal{O}\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)\Bigg),
\end{aligned} \tag{2.21}
$$

*and*

$$
\begin{aligned}
z\int_{\mathbb{R}} \frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} & \frac{1}{\mathbb{E}}\exp\Big(-n(g^o(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+)\Big) \\
& \times\Bigg((\overset{o}{m}{}^{\infty}(z))_{12}\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{11} + \mathcal{O}\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg) \\
& + (\overset{o}{m}{}^{\infty}(z))_{22}\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{12} + \mathcal{O}\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)\Bigg),
\end{aligned} \tag{2.22}
$$

*where*

$$
\begin{aligned}
\mathcal{R}_0^o(z) := \sum_{j=1}^{N+1}\Bigg( & \frac{(\mathcal{B}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o) - \mathcal{A}^o(a_j^o)(\widehat{\alpha}_1^o(a_j^o) + (a_j^o)^{-1}\widehat{\alpha}_0^o(a_j^o)))}{(\widehat{\alpha}_0^o(a_j^o))^2 a_j^o} \\
& + \frac{(\mathcal{B}^o(b_{j-1}^o)\widehat{\alpha}_0^o(b_{j-1}^o) - \mathcal{A}^o(b_{j-1}^o)(\widehat{\alpha}_1^o(b_{j-1}^o) + (b_{j-1}^o)^{-1}\widehat{\alpha}_0^o(b_{j-1}^o)))}{(\widehat{\alpha}_0^o(b_{j-1}^o))^2 b_{j-1}^o} \\
& + \frac{1}{(z-b_{j-1}^o)}\Bigg(\frac{\mathcal{A}^o(b_{j-1}^o)}{\widehat{\alpha}_0^o(b_{j-1}^o)(z-b_{j-1}^o)} + \frac{(\mathcal{B}^o(b_{j-1}^o)\widehat{\alpha}_0^o(b_{j-1}^o) - \mathcal{A}^o(b_{j-1}^o)\widehat{\alpha}_1^o(b_{j-1}^o))}{(\widehat{\alpha}_0^o(b_{j-1}^o))^2}\Bigg) \\
& + \frac{1}{(z-a_j^o)}\Bigg(\frac{\mathcal{A}^o(a_j^o)}{\widehat{\alpha}_0^o(a_j^o)(z-a_j^o)} + \frac{(\mathcal{B}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o) - \mathcal{A}^o(a_j^o)\widehat{\alpha}_1^o(a_j^o))}{(\widehat{\alpha}_0^o(a_j^o))^2}\Bigg)\Bigg),
\end{aligned} \tag{2.23}
$$

*with, for* $j=1,\dots,N+1$,

$$
\mathcal{A}^o(b_{j-1}^o) = -\frac{s_1(\gamma^o(0))^2\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_{j-1}^o}}{Q_0^o(b_{j-1}^o)}\begin{pmatrix} \varkappa_1^o(b_{j-1}^o)\varkappa_2^o(b_{j-1}^o) & \mathrm{i}(\varkappa_1^o(b_{j-1}^o))^2 \\ \mathrm{i}(\varkappa_2^o(b_{j-1}^o))^2 & -\varkappa_1^o(b_{j-1}^o)\varkappa_2^o(b_{j-1}^o) \end{pmatrix}, \tag{2.24}
$$

$$
\mathcal{A}^o(a_j^o) = \frac{s_1 Q_0^o(a_j^o)\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_j^o}}{(\gamma^o(0))^2}\begin{pmatrix} -\varkappa_1^o(a_j^o)\varkappa_2^o(a_j^o) & \mathrm{i}(\varkappa_1^o(a_j^o))^2 \\ \mathrm{i}(\varkappa_2^o(a_j^o))^2 & \varkappa_1^o(a_j^o)\varkappa_2^o(a_j^o) \end{pmatrix}, \tag{2.25}
$$

$$
\frac{\mathcal{B}^o(b_{j-1}^o)}{\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_{j-1}^o}} =
\begin{pmatrix}
\boxed{\begin{matrix}
\varkappa_1^o(b_{j-1}^o)\varkappa_2^o(b_{j-1}^o)\Big(-\frac{s_1(\gamma^o(0))^2}{Q_0^o(b_{j-1}^o)} \\
\times\Big\{\daleth_1^1(b_{j-1}^o) + \daleth_{-1}^1(b_{j-1}^o) - Q_1^o(b_{j-1}^o) \\
\times(Q_0^o(b_{j-1}^o))^{-1}\Big\} - t_1(\gamma^o(0))^2\Big\{Q_0^o(b_{j-1}^o) \\
+ (Q_0^o(b_{j-1}^o))^{-1}\boldsymbol{\aleph}_1^1(b_{j-1}^o)\boldsymbol{\aleph}_{-1}^1(b_{j-1}^o)\Big\} \\
+ \mathrm{i}(s_1+t_1)\Big\{\boldsymbol{\aleph}_{-1}^1(b_{j-1}^o) - \boldsymbol{\aleph}_1^1(b_{j-1}^o)\Big\}\Big)
\end{matrix}} &
\boxed{\begin{matrix}
(\varkappa_1^o(b_{j-1}^o))^2\Big(-\frac{\mathrm{i}s_1(\gamma^o(0))^2}{Q_0^o(b_{j-1}^o)}\Big\{2\daleth_1^1(b_{j-1}^o) \\
- Q_1^o(b_{j-1}^o)(Q_0^o(b_{j-1}^o))^{-1}\Big\} + \mathrm{i}t_1\Big\{Q_0^o(b_{j-1}^o) \\
\times(\gamma^o(0))^{-2} - (Q_0^o(b_{j-1}^o))^{-1}(\gamma^o(0))^2 \\
\times(\boldsymbol{\aleph}_1^1(b_{j-1}^o))^2\Big\} + 2(s_1-t_1)\boldsymbol{\aleph}_1^1(b_{j-1}^o)\Big)
\end{matrix}} \\
\boxed{\begin{matrix}
(\varkappa_2^o(b_{j-1}^o))^2\Big(-\frac{\mathrm{i}s_1(\gamma^o(0))^2}{Q_0^o(b_{j-1}^o)}\Big\{2\daleth_{-1}^1(b_{j-1}^o) \\
- Q_1^o(b_{j-1}^o)(Q_0^o(b_{j-1}^o))^{-1}\Big\} + \mathrm{i}t_1\Big\{Q_0^o(b_{j-1}^o) \\
\times(\gamma^o(0))^{-2} - (Q_0^o(b_{j-1}^o))^{-1}(\gamma^o(0))^2 \\
\times(\boldsymbol{\aleph}_{-1}^1(b_{j-1}^o))^2\Big\} - 2(s_1-t_1)\boldsymbol{\aleph}_{-1}^1(b_{j-1}^o)\Big)
\end{matrix}} &
\boxed{\begin{matrix}
\varkappa_1^o(b_{j-1}^o)\varkappa_2^o(b_{j-1}^o)\Big(\frac{s_1(\gamma^o(0))^2}{Q_0^o(b_{j-1}^o)} \\
\times\Big\{\daleth_1^1(b_{j-1}^o) + \daleth_{-1}^1(b_{j-1}^o) - Q_1^o(b_{j-1}^o) \\
\times(Q_0^o(b_{j-1}^o))^{-1}\Big\} + t_1(\gamma^o(0))^2\Big\{Q_0^o(b_{j-1}^o) \\
+ (Q_0^o(b_{j-1}^o))^{-1}\boldsymbol{\aleph}_1^1(b_{j-1}^o)\boldsymbol{\aleph}_{-1}^1(b_{j-1}^o)\Big\} \\
+ \mathrm{i}(s_1+t_1)\Big\{\boldsymbol{\aleph}_1^1(b_{j-1}^o) - \boldsymbol{\aleph}_{-1}^1(b_{j-1}^o)\Big\}\Big)
\end{matrix}}
\end{pmatrix}, \tag{2.26}
$$

$$\frac{\mathcal{B}^o(a_j^o)}{\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_j^o}} = \begin{pmatrix} \boxed{\begin{gathered} \varkappa_1^o(a_j^o)\varkappa_2^o(a_j^o)\Big(-\tfrac{s_1}{(\gamma^o(0))^2}\Big\{Q_1^o(a_j^o) \\ +\,Q_0^o(a_j^o)\left[\daleth_1^1(a_j^o)+\daleth_{-1}^1(a_j^o)\right]\Big\}-t_1 \\ \times\Big\{(\gamma^o(0))^{-2}Q_0^o(a_j^o)\boldsymbol{\aleph}_1^1(a_j^o)\boldsymbol{\aleph}_{-1}^1(a_j^o) \\ +\,(\gamma^o(0))^2(Q_0^o(a_j^o))^{-1}\Big\} \\ +\,\mathrm{i}(s_1+t_1)\Big\{\boldsymbol{\aleph}_{-1}^1(a_j^o)-\boldsymbol{\aleph}_1^1(a_j^o)\Big\}\Big) \end{gathered}} & \boxed{\begin{gathered} (\varkappa_1^o(a_j^o))^2\Big(\tfrac{\mathrm{i}s_1}{(\gamma^o(0))^2}\Big\{Q_1^o(a_j^o)+2Q_0^o(a_j^o) \\ \times\daleth_1^1(a_j^o)\Big\}+\mathrm{i}t_1\Big\{Q_0^o(a_j^o)(\boldsymbol{\aleph}_1^1(a_j^o))^2 \\ \times(\gamma^o(0))^{-2}-(Q_0^o(a_j^o))^{-1}(\gamma^o(0))^2\Big\} \\ -\,2(s_1-t_1)\boldsymbol{\aleph}_1^1(a_j^o)\Big) \end{gathered}} \\ \boxed{\begin{gathered} (\varkappa_2^o(a_j^o))^2\Big(\tfrac{\mathrm{i}s_1}{(\gamma^o(0))^2}\Big\{Q_1^o(a_j^o)+2Q_0^o(a_j^o) \\ \times\daleth_{-1}^1(a_j^o)\Big\}+\mathrm{i}t_1\Big\{Q_0^o(a_j^o)(\boldsymbol{\aleph}_{-1}^1(a_j^o))^2 \\ \times(\gamma^o(0))^{-2}-(Q_0^o(a_j^o))^{-1}(\gamma^o(0))^2\Big\} \\ +\,2(s_1-t_1)\boldsymbol{\aleph}_{-1}^1(a_j^o)\Big) \end{gathered}} & \boxed{\begin{gathered} \varkappa_1^o(a_j^o)\varkappa_2^o(a_j^o)\Big(\tfrac{s_1}{(\gamma^o(0))^2}\Big\{Q_1^o(a_j^o) \\ +\,Q_0^o(a_j^o)\left[\daleth_1^1(a_j^o)+\daleth_{-1}^1(a_j^o)\right]\Big\}+t_1 \\ \times\Big\{(\gamma^o(0))^{-2}Q_0^o(a_j^o)\boldsymbol{\aleph}_1^1(a_j^o)\boldsymbol{\aleph}_{-1}^1(a_j^o) \\ +\,(\gamma^o(0))^2(Q_0^o(a_j^o))^{-1}\Big\} \\ +\,\mathrm{i}(s_1+t_1)\Big\{\boldsymbol{\aleph}_1^1(a_j^o)-\boldsymbol{\aleph}_{-1}^1(a_j^o)\Big\}\Big) \end{gathered}} \end{pmatrix}, \tag{2.27}$$

$$s_1=\frac{5}{72}, \qquad t_1=-\frac{7}{72}, \qquad \mho_i^o:=\begin{cases}\Omega_i^o, & i=1,\dots,N, \\ 0, & i=0,N+1,\end{cases} \tag{2.28}$$

$$Q_0^o(b_0^o)=-\mathrm{i}\left((a_{N+1}^o-b_0^o)^{-1}\prod_{k=1}^{N}\left(\frac{b_k^o-b_0^o}{a_k^o-b_0^o}\right)\right)^{1/2}, \tag{2.29}$$

$$Q_1^o(b_0^o)=\frac{1}{2}Q_0^o(b_0^o)\left(\sum_{k=1}^{N}\left(\frac{1}{b_0^o-b_k^o}-\frac{1}{b_0^o-a_k^o}\right)-\frac{1}{b_0^o-a_{N+1}^o}\right), \tag{2.30}$$

$$Q_0^o(a_{N+1}^o)=\left((a_{N+1}^o-b_0^o)\prod_{k=1}^{N}\left(\frac{a_{N+1}^o-b_k^o}{a_{N+1}^o-a_k^o}\right)\right)^{1/2}, \tag{2.31}$$

$$Q_1^o(a_{N+1}^o)=\frac{1}{2}Q_0^o(a_{N+1}^o)\left(\sum_{k=1}^{N}\left(\frac{1}{a_{N+1}^o-b_k^o}-\frac{1}{a_{N+1}^o-a_k^o}\right)+\frac{1}{a_{N+1}^o-b_0^o}\right), \tag{2.32}$$

$$Q_0^o(b_j^o)=-\mathrm{i}\left(\frac{(b_j^o-b_0^o)}{(a_{N+1}^o-b_j^o)(b_j^o-a_j^o)}\prod_{k=1}^{j-1}\left(\frac{b_j^o-b_k^o}{b_j^o-a_k^o}\right)\prod_{l=j+1}^{N}\left(\frac{b_l^o-b_j^o}{a_l^o-b_j^o}\right)\right)^{1/2}, \tag{2.33}$$

$$Q_1^o(b_j^o)=\frac{1}{2}Q_0^o(b_j^o)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{b_j^o-b_k^o}-\frac{1}{b_j^o-a_k^o}\right)+\frac{1}{b_j^o-b_0^o}-\frac{1}{b_j^o-a_{N+1}^o}-\frac{1}{b_j^o-a_j^o}\right), \tag{2.34}$$

$$Q_0^o(a_j^o)=\left(\frac{(a_j^o-b_0^o)(b_j^o-a_j^o)}{(a_{N+1}^o-a_j^o)}\prod_{k=1}^{j-1}\left(\frac{a_j^o-b_k^o}{a_j^o-a_k^o}\right)\prod_{l=j+1}^{N}\left(\frac{b_l^o-a_j^o}{a_l^o-a_j^o}\right)\right)^{1/2}, \tag{2.35}$$

$$Q_1^o(a_j^o)=\frac{1}{2}Q_0^o(a_j^o)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{a_j^o-b_k^o}-\frac{1}{a_j^o-a_k^o}\right)+\frac{1}{a_j^o-b_0^o}-\frac{1}{a_j^o-a_{N+1}^o}+\frac{1}{a_j^o-b_j^o}\right), \tag{2.36}$$

*where* $\mathrm{i}Q_0^o(b_{j-1}^o), Q_0^o(a_j^o)>0$, $j=1,\dots,N+1$,

$$\varkappa_1^o(\xi)=\frac{1}{\mathbb{E}}\frac{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)+\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(\xi)+\boldsymbol{d}_o)}, \tag{2.37}$$

$$\varkappa_2^o(\xi)=\mathbb{E}\frac{\boldsymbol{\theta}^o(-\boldsymbol{u}_+^o(0)-\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(-\boldsymbol{u}_+^o(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(\xi)-\boldsymbol{d}_o)}, \tag{2.38}$$

$$\boldsymbol{\aleph}_{\varepsilon_2}^{\varepsilon_1}(\xi)=-\frac{\mathfrak{u}^o(\varepsilon_1,\varepsilon_2,\boldsymbol{0};\xi)}{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}_+^o(\xi)+\varepsilon_2\boldsymbol{d}_o)}+\frac{\mathfrak{u}^o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o;\xi)}{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}_+^o(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)}, \quad \varepsilon_1,\varepsilon_2=\pm1, \tag{2.39}$$

$$
\begin{aligned}
\daleth^{\varepsilon_1}_{\varepsilon_2}(\xi) = &-\frac{\mathfrak{v}^o(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o_+(\xi)+\varepsilon_2\boldsymbol{d}_o)}+\frac{\mathfrak{v}^o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o;\xi)}{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o_+(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)}-\left(\frac{\mathfrak{u}^o(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o_+(\xi)+\varepsilon_2\boldsymbol{d}_o)}\right)^2 \\
&+\frac{\mathfrak{u}^o(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o;\xi)}{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o_+(\xi)+\varepsilon_2\boldsymbol{d}_o)\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o_+(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)},
\end{aligned}
\tag{2.40}
$$

$$\mathfrak{u}^o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o,\xi):=2\pi\Lambda^1_o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o,\xi),\qquad \mathfrak{v}^o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o,\xi):=-2\pi^2\Lambda^2_o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o,\xi), \tag{2.41}$$

$$\Lambda^{j'}_o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o,\xi)=\sum_{m\in\mathbb{Z}^N}(\mathfrak{r}_o(\xi))^{j'}\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}^o_+(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)+\pi\mathrm{i}(m,\boldsymbol{\tau}^o m)},\quad j'=1,2, \tag{2.42}$$

$$\mathfrak{r}_o(\xi):=\frac{2(m,\vec{\mathfrak{z}}_o(\xi))}{\rightthreetimes^o(\xi)},\qquad \vec{\mathfrak{z}}_o(\xi)=(\leftthreetimes^o_1(\xi),\leftthreetimes^o_2(\xi),\ldots,\leftthreetimes^o_N(\xi)), \tag{2.43}$$

$$\leftthreetimes^o_{j'}(\xi):=\sum_{k=1}^N c^o_{j'k}\xi^{N-k},\quad j'=1,\ldots,N, \tag{2.44}$$

$$\rightthreetimes^o(b^o_0)=\mathrm{i}(-1)^N\eta_{b^o_0},\quad \rightthreetimes^o(a^o_{N+1})=\eta_{a^o_{N+1}},\quad \rightthreetimes^o(b^o_j)=\mathrm{i}(-1)^{N-j}\eta_{b^o_j},\quad \rightthreetimes^o(a^o_j)=(-1)^{N-j+1}\eta_{a^o_j}, \tag{2.45}$$

$$\eta_{b^o_0}:=\left((a^o_{N+1}-b^o_0)\prod_{k=1}^N(b^o_k-b^o_0)(a^o_k-b^o_0)\right)^{1/2}, \tag{2.46}$$

$$\eta_{a^o_{N+1}}:=\left((a^o_{N+1}-b^o_0)\prod_{k=1}^N(a^o_{N+1}-b^o_k)(a^o_{N+1}-a^o_k)\right)^{1/2}, \tag{2.47}$$

$$\eta_{b^o_j}:=\left((b^o_j-a^o_j)(a^o_{N+1}-b^o_j)(b^o_j-b^o_0)\prod_{k=1}^{j-1}(b^o_j-b^o_k)(b^o_j-a^o_k)\prod_{l=j+1}^N(b^o_l-b^o_j)(a^o_l-b^o_j)\right)^{1/2}, \tag{2.48}$$

$$\eta_{a^o_j}:=\left((b^o_j-a^o_j)(a^o_{N+1}-a^o_j)(a^o_j-b^o_0)\prod_{k=1}^{j-1}(a^o_j-b^o_k)(a^o_j-a^o_k)\prod_{l=j+1}^N(b^o_l-a^o_j)(a^o_l-a^o_j)\right)^{1/2}, \tag{2.49}$$

*where* $c^o_{j'k'}$, $j',k'=1,\ldots,N$, *are obtained from Equations* (O1) *and* (O2), $\eta_{b^o_{j-1}},\eta_{a^o_j}>0$, $j=1,\ldots,N+1$, *and*

$$\widehat{\alpha}^o_0(b^o_0)=\frac{4}{3}\mathrm{i}(-1)^N h^o_V(b^o_0)\eta_{b^o_0}, \tag{2.50}$$

$$\widehat{\alpha}^o_1(b^o_0)=\mathrm{i}(-1)^N\left(\frac{2}{5}h^o_V(b^o_0)\eta_{b^o_0}\left(\sum_{l=1}^N\left(\frac{1}{b^o_0-b^o_l}+\frac{1}{b^o_0-a^o_l}\right)+\frac{1}{b^o_0-a^o_{N+1}}\right)+\frac{4}{5}(h^o_V(b^o_0))'\eta_{b^o_0}\right), \tag{2.51}$$

$$\widehat{\alpha}^o_0(a^o_{N+1})=\frac{4}{3}h^o_V(a^o_{N+1})\eta_{a^o_{N+1}}, \tag{2.52}$$

$$\widehat{\alpha}^o_1(a^o_{N+1})=\frac{2}{5}h^o_V(a^o_{N+1})\eta_{a^o_{N+1}}\left(\sum_{l=1}^N\left(\frac{1}{a^o_{N+1}-b^o_l}+\frac{1}{a^o_{N+1}-a^o_l}\right)+\frac{1}{a^o_{N+1}-b^o_0}\right)+\frac{4}{5}(h^o_V(a^o_{N+1}))'\eta_{a^o_{N+1}}, \tag{2.53}$$

$$\widehat{\alpha}^o_0(b^o_j)=\frac{4}{3}\mathrm{i}(-1)^{N-j}h^o_V(b^o_j)\eta_{b^o_j}, \tag{2.54}$$

$$\widehat{\alpha}^o_1(b^o_j)=\mathrm{i}(-1)^{N-j}\left(\frac{2}{5}h^o_V(b^o_j)\eta_{b^o_j}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{b^o_j-b^o_k}+\frac{1}{b^o_j-a^o_k}\right)+\frac{1}{b^o_j-a^o_j}+\frac{1}{b^o_j-a^o_{N+1}}+\frac{1}{b^o_j-b^o_0}\right)+\frac{4}{5}(h^o_V(b^o_j))'\eta_{b^o_j}\right), \tag{2.55}$$

$$\widehat{\alpha}^o_0(a^o_j)=\frac{4}{3}(-1)^{N-j+1}h^o_V(a^o_j)\eta_{a^o_j}, \tag{2.56}$$

$$\widehat{\alpha}^o_1(a^o_j)=(-1)^{N-j+1}\left(\frac{2}{5}h^o_V(a^o_j)\eta_{a^o_j}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{a^o_j-b^o_k}+\frac{1}{a^o_j-a^o_k}\right)+\frac{1}{a^o_j-b^o_j}+\frac{1}{a^o_j-a^o_{N+1}}+\frac{1}{a^o_j-b^o_0}\right)+\frac{4}{5}(h^o_V(a^o_j))'\eta_{a^o_j}\right); \tag{2.57}$$

**(2)** *for* $z \in \Upsilon_2^o$ $(\subset \mathbb{C}_-)$,

$$z\boldsymbol{\pi}_{2n+1}(z) \underset{n\to\infty}{=} \frac{1}{\mathbb{E}} \exp\Big(n(g^o(z)-\mathfrak{Q}_{\mathcal{A}}^-)\Big)\Bigg((\overset{o}{m}{}^{\infty}(z))_{11}\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{11} + O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)+(\overset{o}{m}{}^{\infty}(z))_{21}\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{12}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)\Bigg), \tag{2.58}$$

*and*

$$z\int_{\mathbb{R}} \frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \mathbb{E}\exp\Big(-n(g^o(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^-)\Big) \times\Bigg((\overset{o}{m}{}^{\infty}(z))_{12}\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{11}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg) + (\overset{o}{m}{}^{\infty}(z))_{22}\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{12}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)\Bigg); \tag{2.59}$$

**(3)** *for* $z \in \Upsilon_3^o$ $(\subset \cup_{j=1}^{N+1}\Big\{z\in\mathbb{C}^*;\ \mathrm{Re}(z)\in(b_{j-1}^o,a_j^o),\ \inf_{q\in(b_{j-1}^o,a_j^o)}|z-q|<2^{-1/2}\min\{\delta_{b_{j-1}}^o,\delta_{a_j}^o\}\Big\}\subset\mathbb{C}_+)$,

$$z\boldsymbol{\pi}_{2n+1}(z) \underset{n\to\infty}{=} \mathbb{E}\exp\Big(n(g^o(z)-\mathfrak{Q}_{\mathcal{A}}^+)\Big)\Bigg(\Big((\overset{o}{m}{}^{\infty}(z))_{11}+(\overset{o}{m}{}^{\infty}(z))_{12}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\Big) \times\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{11}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)+\Big((\overset{o}{m}{}^{\infty}(z))_{22}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s} + (\overset{o}{m}{}^{\infty}(z))_{21}\Big)\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{12}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)\Bigg), \tag{2.60}$$

*and*

$$z\int_{\mathbb{R}} \frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \frac{1}{\mathbb{E}}\exp\Big(-n(g^o(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+)\Big)\Big((\overset{o}{m}{}^{\infty}(z))_{12} \times\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{11}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)+(\overset{o}{m}{}^{\infty}(z))_{22} \times\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{12}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)\Bigg); \tag{2.61}$$

**(4)** *for* $z \in \Upsilon_4^o$ $(\subset \cup_{j=1}^{N+1}\Big\{z\in\mathbb{C}^*;\ \mathrm{Re}(z)\in(b_{j-1}^o,a_j^o),\ \inf_{q\in(b_{j-1}^o,a_j^o)}|z-q|<2^{-1/2}\min\{\delta_{b_{j-1}}^o,\delta_{a_j}^o\}\Big\}\subset\mathbb{C}_-)$,

$$z\boldsymbol{\pi}_{2n+1}(z) \underset{n\to\infty}{=} \frac{1}{\mathbb{E}}\exp\Big(n(g^o(z)-\mathfrak{Q}_{\mathcal{A}}^-)\Big)\Bigg(\Big((\overset{o}{m}{}^{\infty}(z))_{11}-(\overset{o}{m}{}^{\infty}(z))_{12}\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\Big) \times\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{11}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)+\Big(-(\overset{o}{m}{}^{\infty}(z))_{22}\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s} + (\overset{o}{m}{}^{\infty}(z))_{21}\Big)\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{12}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)\Bigg), \tag{2.62}$$

*and*

$$z\int_{\mathbb{R}} \frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \mathbb{E}\exp\Big(-n(g^o(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^-)\Big)\Big((\overset{o}{m}{}^{\infty}(z))_{12} \times\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{11}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)+(\overset{o}{m}{}^{\infty}(z))_{22} \times\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)\Big)_{12}+O\Bigg(\frac{1}{(n+\frac{1}{2})^2}\Bigg)\Bigg)\Bigg); \tag{2.63}$$

**(5)** *for* $z \in \Omega^{o,1}_{b_{j-1}}$ $(\subset \mathbb{C}_{+} \cap \mathbb{U}^{o}_{\delta_{b_{j-1}}})$, $j=1,\dots,N+1$,

$$z\boldsymbol{\pi}_{2n+1}(z) \underset{n\to\infty}{=} \mathbb{E}\exp\Big(n(g^{o}(z)-\mathfrak{Q}^{+}_{\mathcal{A}})\Big)\Bigg((m^{b,1}_{p}(z))_{11}\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{11} + O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\Bigg)+(m^{b,1}_{p}(z))_{21}\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{12}+O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\Bigg)\Bigg), \tag{2.64}$$

*and*

$$z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \frac{1}{\mathbb{E}}\exp\Big(-n(g^{o}(z)-\ell_{o}-\mathfrak{Q}^{+}_{\mathcal{A}})\Big)\Bigg((m^{b,1}_{p}(z))_{12} \times\Bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{11}+O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\Bigg)+(m^{b,1}_{p}(z))_{22} \times\Bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{12}+O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\Bigg)\Bigg), \tag{2.65}$$

*where*

$$(m^{b,1}_{p}(z))_{11} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_b)(p_b)^{1/4}-\mathrm{Ai}'(p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11} - \Big(\mathrm{Ai}(p_b)(p_b)^{1/4}+\mathrm{Ai}'(p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}\Big), \tag{2.66}$$

$$(m^{b,1}_{p}(z))_{12} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\mathrm{Ai}(\omega^{2}p_b)(p_b)^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11} \times\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}+\Big(\mathrm{Ai}(\omega^{2}p_b)(p_b)^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\Big), \tag{2.67}$$

$$(m^{b,1}_{p}(z))_{21} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_b)(p_b)^{1/4}-\mathrm{Ai}'(p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21} - \Big(\mathrm{Ai}(p_b)(p_b)^{1/4}+\mathrm{Ai}'(p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}\Big), \tag{2.68}$$

$$(m^{b,1}_{p}(z))_{22} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\mathrm{Ai}(\omega^{2}p_b)(p_b)^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21} \times\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}+\Big(\mathrm{Ai}(\omega^{2}p_b)(p_b)^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\Big), \tag{2.69}$$

*with* $\omega=\exp(2\pi\mathrm{i}/3)$,

$$\widetilde{\mathcal{R}}^{o}_{0}(z) := \sum_{j=1}^{N+1}\Big(\mathcal{R}^{0}_{b^{o}_{j-1}}(z)\mathbf{1}_{\mathbb{U}^{o}_{\delta_{b_{j-1}}}}(z)+\mathcal{R}^{0}_{a^{o}_{j}}(z)\mathbf{1}_{\mathbb{U}^{o}_{\delta_{a_j}}}(z)\Big), \tag{2.70}$$

$$\xi^{o}_{b_{j-1}}(z)=-2\int_{z}^{b^{o}_{j-1}}(R_{o}(s))^{1/2}h^{o}_{V}(s)\,\mathrm{d}s, \qquad p_b=\bigg(\frac{3}{4}\Big(n+\frac{1}{2}\Big)\xi^{o}_{b_{j-1}}(z)\bigg)^{2/3}, \tag{2.71}$$

$$\mathcal{R}^{0}_{b^{o}_{j-1}}(z)=\frac{1}{\xi^{o}_{b_{j-1}}(z)}\overset{o}{m}{}^{\infty}(z)\begin{pmatrix} -(s_1+t_1) & -\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}} \\ -\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}} & (s_1+t_1)\end{pmatrix}(\overset{o}{m}{}^{\infty}(z))^{-1}, \tag{2.72}$$

$$\mathcal{R}^{0}_{a^{o}_{j}}(z)=\frac{1}{\xi^{o}_{a_j}(z)}\overset{o}{m}{}^{\infty}(z)\begin{pmatrix} -(s_1+t_1) & \mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}} \\ \mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}} & (s_1+t_1)\end{pmatrix}(\overset{o}{m}{}^{\infty}(z))^{-1}, \tag{2.73}$$

$$\xi^{o}_{a_j}(z)=2\int_{a^{o}_{j}}^{z}(R_{o}(s))^{1/2}h^{o}_{V}(s)\,\mathrm{d}s, \tag{2.74}$$

*and* $\mathbf{1}_{\mathbb{U}^{o}_{\delta_{b_{j-1}}}}(z)$ *(resp.,* $\mathbf{1}_{\mathbb{U}^{o}_{\delta_{a_j}}}(z)$*) the indicator (characteristic) function of the set* $\mathbb{U}^{o}_{\delta_{b_{j-1}}}$ *(resp.,* $\mathbb{U}^{o}_{\delta_{a_j}}$*);*
**(6)** *for* $z \in \Omega^{o,2}_{b_{j-1}}$ $(\subset \mathbb{C}_{+} \cap \mathbb{U}^{o}_{\delta_{b_{j-1}}})$, $j=1,\dots,N+1$,

$$z\boldsymbol{\pi}_{2n+1}(z) \underset{n\to\infty}{=} \mathbb{E}\exp\Big(n(g^{o}(z)-\mathfrak{Q}^{+}_{\mathcal{A}})\Big)\bigg(\Big((m^{b,2}_{p}(z))_{11}+(m^{b,2}_{p}(z))_{12}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_{z}^{a^{o}_{N+1}}\psi^{o}_{V}(s)\,\mathrm{d}s}\Big)$$

$$\times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)+\Big((m_p^{b,2}(z))_{21}+(m_p^{b,2}(z))_{22}$$
$$\times\ \mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\Big)\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)\Bigg), \tag{2.75}$$

*and*

$$z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\frac{1}{\mathbb{E}}\exp\Big(-n(g^o(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+)\Big)\Big((m_p^{b,2}(z))_{12}$$
$$\times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)+(m_p^{b,2}(z))_{22}$$
$$\times\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)\Bigg), \tag{2.76}$$

*where*

$$(m_p^{b,2}(z))_{11}:=-\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi_{b_{j-1}}^o(z)}\Big(\mathrm{i}\Big(-\omega\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{11}$$
$$+\Big(\omega\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{12}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho_{j-1}^o}\Big), \tag{2.77}$$

$$(m_p^{b,2}(z))_{12}:=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi_{b_{j-1}}^o(z)}\Big(\mathrm{i}\Big(-\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{11}$$
$$\times\,\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_{j-1}^o}+\Big(\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{12}\Big), \tag{2.78}$$

$$(m_p^{b,2}(z))_{21}:=-\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi_{b_{j-1}}^o(z)}\Big(\mathrm{i}\Big(-\omega\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{21}$$
$$+\Big(\omega\,\mathrm{Ai}(\omega p_b)(p_b)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{22}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho_{j-1}^o}\Big), \tag{2.79}$$

$$(m_p^{b,2}(z))_{22}:=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi_{b_{j-1}}^o(z)}\Big(\mathrm{i}\Big(-\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{21}$$
$$\times\,\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_{j-1}^o}+\Big(\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{22}\Big); \tag{2.80}$$

**(7)** *for* $z\in\Omega_{b_{j-1}}^{o,3}$ $(\subset\mathbb{C}_-\cap\mathbb{U}_{\delta_{b_{j-1}}}^o)$, $j=1,\dots,N+1$,

$$z\boldsymbol{\pi}_{2n+1}(z)\underset{n\to\infty}{=}\frac{1}{\mathbb{E}}\exp\Big(n(g^o(z)-\mathfrak{Q}_{\mathcal{A}}^-)\Big)\Bigg(\Big((m_p^{b,3}(z))_{11}-(m_p^{b,3}(z))_{12}\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\Big)$$
$$\times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)+\Big((m_p^{b,3}(z))_{21}-(m_p^{b,3}(z))_{22}$$
$$\times\ \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\Big)\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)\Bigg), \tag{2.81}$$

*and*

$$z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\mathbb{E}\exp\Big(-n(g^o(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^-)\Big)\Big((m_p^{b,3}(z))_{12}$$
$$\times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)+(m_p^{b,3}(z))_{22}$$
$$\times\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)\Bigg), \tag{2.82}$$

*where*

$$(m_p^{b,3}(z))_{11}:=-\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi_{b_{j-1}}^o(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega^2 p_b)(p_b)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2 p_b)(p_b)^{-1/4}\Big)(\overset{o}{m}{}^\infty(z))_{11}$$

$$+\Big(\omega^{2}\,\mathrm{Ai}(\omega^{2}p_{b})(p_{b})^{1/4}+\omega\,\mathrm{Ai}'(\omega^{2}p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{12}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}\Big), \tag{2.83}$$

$$\begin{aligned}(m^{b,3}_{p}(z))_{12} :=\ &\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\omega^{2}\,\mathrm{Ai}(\omega p_{b})(p_{b})^{1/4}-\mathrm{Ai}'(\omega p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{11}\\ &\times\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}-\Big(\omega^{2}\,\mathrm{Ai}(\omega p_{b})(p_{b})^{1/4}+\mathrm{Ai}'(\omega p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{12}\Big),\end{aligned} \tag{2.84}$$

$$\begin{aligned}(m^{b,3}_{p}(z))_{21} :=\ &-\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(-\omega^{2}\,\mathrm{Ai}(\omega^{2}p_{b})(p_{b})^{1/4}+\omega\,\mathrm{Ai}'(\omega^{2}p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{21}\\ &+\Big(\omega^{2}\,\mathrm{Ai}(\omega^{2}p_{b})(p_{b})^{1/4}+\omega\,\mathrm{Ai}'(\omega^{2}p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{22}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}\Big),\end{aligned} \tag{2.85}$$

$$\begin{aligned}(m^{b,3}_{p}(z))_{22} :=\ &\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\omega^{2}\,\mathrm{Ai}(\omega p_{b})(p_{b})^{1/4}-\mathrm{Ai}'(\omega p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{21}\\ &\times\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}-\Big(\omega^{2}\,\mathrm{Ai}(\omega p_{b})(p_{b})^{1/4}+\mathrm{Ai}'(\omega p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{22}\Big);\end{aligned} \tag{2.86}$$

**(8)** *for* $z\in\Omega^{o,4}_{b_{j-1}}$ $(\subset\mathbb{C}_{-}\cap\mathbb{U}^{o}_{\delta_{b_{j-1}}})$, $j=1,\dots,N+1$,

$$\begin{aligned}z\boldsymbol{\pi}_{2n+1}(z) \underset{n\to\infty}{=}\ &\frac{1}{\mathbb{E}}\exp\Big(n(g^{o}(z)-\mathfrak{Q}^{-}_{\mathcal{A}})\Big)\bigg((m^{b,4}_{p}(z))_{11}\bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{11}\\ &+\ O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\bigg)+(m^{b,4}_{p}(z))_{21}\bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{12}+O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\bigg)\bigg),\end{aligned} \tag{2.87}$$

*and*

$$\begin{aligned}z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=}\ &\mathbb{E}\exp\Big(-n(g^{o}(z)-\ell_{o}-\mathfrak{Q}^{-}_{\mathcal{A}})\Big)\Big((m^{b,4}_{p}(z))_{12}\\ &\times\bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{11}+O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\bigg)+(m^{b,4}_{p}(z))_{22}\\ &\times\bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{12}+O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\bigg)\Big),\end{aligned} \tag{2.88}$$

*where*

$$\begin{aligned}(m^{b,4}_{p}(z))_{11} :=\ &-\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_{b})(p_{b})^{1/4}-\mathrm{Ai}'(p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{11}\\ &-\Big(\mathrm{Ai}(p_{b})(p_{b})^{1/4}+\mathrm{Ai}'(p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{12}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}\Big),\end{aligned} \tag{2.89}$$

$$\begin{aligned}(m^{b,4}_{p}(z))_{12} :=\ &\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\omega^{2}\,\mathrm{Ai}(\omega p_{b})(p_{b})^{1/4}-\mathrm{Ai}'(\omega p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{11}\\ &\times\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}-\Big(\omega^{2}\,\mathrm{Ai}(\omega p_{b})(p_{b})^{1/4}+\mathrm{Ai}'(\omega p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{12}\Big),\end{aligned} \tag{2.90}$$

$$\begin{aligned}(m^{b,4}_{p}(z))_{21} :=\ &-\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_{b})(p_{b})^{1/4}-\mathrm{Ai}'(p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{21}\\ &-\Big(\mathrm{Ai}(p_{b})(p_{b})^{1/4}+\mathrm{Ai}'(p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{22}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}\Big),\end{aligned} \tag{2.91}$$

$$\begin{aligned}(m^{b,4}_{p}(z))_{22} :=\ &\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\Big(\mathrm{i}\Big(\omega^{2}\,\mathrm{Ai}(\omega p_{b})(p_{b})^{1/4}-\mathrm{Ai}'(\omega p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{21}\\ &\times\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}-\Big(\omega^{2}\,\mathrm{Ai}(\omega p_{b})(p_{b})^{1/4}+\mathrm{Ai}'(\omega p_{b})(p_{b})^{-1/4}\Big)(\overset{o}{m}^{\infty}(z))_{22}\Big);\end{aligned} \tag{2.92}$$

**(9)** *for* $z\in\Omega^{o,1}_{a_{j}}$ $(\subset\mathbb{C}_{+}\cap\mathbb{U}^{o}_{\delta_{a_{j}}})$, $j=1,\dots,N+1$,

$$\begin{aligned}z\boldsymbol{\pi}_{2n+1}(z) \underset{n\to\infty}{=}\ &\mathbb{E}\exp\Big(n(g^{o}(z)-\mathfrak{Q}^{+}_{\mathcal{A}})\Big)\bigg((m^{a,1}_{p}(z))_{11}\bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{11}\\ &+\ O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\bigg)+(m^{a,1}_{p}(z))_{21}\bigg(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{12}+O\bigg(\frac{1}{(n+\frac{1}{2})^{2}}\bigg)\bigg)\bigg),\end{aligned} \tag{2.93}$$

*and*

$$z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\frac{1}{\mathbb{E}}\exp\Big(-n(g^{o}(z)-\ell_{o}-\mathfrak{Q}^{+}_{\mathcal{A}})\Big)\Big((m^{a,1}_{p}(z))_{12}\\ \times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^{2}}\right)\right)+(m^{a,1}_{p}(z))_{22}\\ \times\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^{2}}\right)\right)\Big), \tag{2.94}$$

*where*

$$(m^{a,1}_{p}(z))_{11}:=-\,\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_{j}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_{a})(p_{a})^{1/4}-\mathrm{Ai}'(p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11}\\ +\Big(\mathrm{Ai}(p_{a})(p_{a})^{1/4}+\mathrm{Ai}'(p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}}\Big), \tag{2.95}$$

$$(m^{a,1}_{p}(z))_{12}:=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_{j}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(\omega^{2}p_{a})(p_{a})^{1/4}-\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11}\\ \times\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}}+\Big(\mathrm{Ai}(\omega^{2}p_{a})(p_{a})^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\Big), \tag{2.96}$$

$$(m^{a,1}_{p}(z))_{21}:=-\,\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_{j}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_{a})(p_{a})^{1/4}-\mathrm{Ai}'(p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21}\\ +\Big(\mathrm{Ai}(p_{a})(p_{a})^{1/4}+\mathrm{Ai}'(p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}}\Big), \tag{2.97}$$

$$(m^{a,1}_{p}(z))_{22}:=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_{j}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(\omega^{2}p_{a})(p_{a})^{1/4}-\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21}\\ \times\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}}+\Big(\mathrm{Ai}(\omega^{2}p_{a})(p_{a})^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\Big), \tag{2.98}$$

*with*

$$p_{a}=\left(\frac{3}{4}\Big(n+\frac{1}{2}\Big)\xi^{o}_{a_{j}}(z)\right)^{2/3}; \tag{2.99}$$

**(10)** *for* $z\in\Omega^{o,2}_{a_{j}}$ $(\subset\mathbb{C}_{+}\cap\mathbb{U}^{o}_{\delta_{a_{j}}})$, $j=1,\dots,N+1$,

$$z\boldsymbol{\pi}_{2n+1}(z)\underset{n\to\infty}{=}\mathbb{E}\exp\Big(n(g^{0}(z)-\mathfrak{Q}^{+}_{\mathcal{A}})\Big)\Big(\Big((m^{a,2}_{p}(z))_{11}+(m^{a,2}_{p}(z))_{12}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_{z}^{a^{o}_{N+1}}\psi^{o}_{V}(s)\,\mathrm{d}s}\Big)\\ \times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^{2}}\right)\right)+\Big((m^{a,2}_{p}(z))_{21}+(m^{a,2}_{p}(z))_{22}\\ \times\;\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_{z}^{a^{o}_{N+1}}\psi^{o}_{V}(s)\,\mathrm{d}s}\Big)\left(\frac{1}{n}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^{2}}\right)\right)\Big), \tag{2.100}$$

*and*

$$z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\frac{1}{\mathbb{E}}\exp\Big(-n(g^{o}(z)-\ell_{o}-\mathfrak{Q}^{+}_{\mathcal{A}})\Big)\Big((m^{a,2}_{p}(z))_{12}\\ \times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^{2}}\right)\right)+(m^{a,2}_{p}(z))_{22}\\ \times\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^{o}_{0}(z)-\widetilde{\mathcal{R}}^{o}_{0}(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^{2}}\right)\right)\Big), \tag{2.101}$$

*where*

$$(m^{a,2}_{p}(z))_{11}:=-\,\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_{j}}(z)}\Big(\mathrm{i}\Big(-\omega\,\mathrm{Ai}(\omega p_{a})(p_{a})^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11}\\ -\Big(\omega\,\mathrm{Ai}(\omega p_{a})(p_{a})^{1/4}+\omega^{2}\,\mathrm{Ai}'(\omega p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}}\Big), \tag{2.102}$$

$$(m^{a,2}_{p}(z))_{12}:=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_{j}}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(\omega^{2}p_{a})(p_{a})^{1/4}-\omega^{2}\,\mathrm{Ai}'(\omega^{2}p_{a})(p_{a})^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11}$$

$$\times\,\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_j^o}+\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\Big),\tag{2.103}$$

$$(m_p^{a,2}(z))_{21}:=-\,\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21}$$
$$-\Big(\omega\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho_j^o}\Big),\tag{2.104}$$

$$(m_p^{a,2}(z))_{22}:=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}-\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21}$$
$$\times\,\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_j^o}+\Big(\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega^2\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\Big);\tag{2.105}$$

**(11)** *for* $z\in\Omega^{o,3}_{a_j}$ $(\subset\mathbb{C}_-\cap\mathbb{U}^o_{\delta_{a_j}})$, $j=1,\dots,N+1$,

$$z\boldsymbol{\pi}_{2n+1}(z)\underset{n\to\infty}{=}\frac{1}{\mathbb{E}}\exp\Big(n(g^o(z)-\mathfrak{Q}^-_{\mathcal{A}})\Big)\bigg(\Big((m_p^{a,3}(z))_{11}-(m_p^{a,3}(z))_{12}\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\Big)$$
$$\times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^o_0(z)-\widetilde{\mathcal{R}}^o_0(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)+\Big((m_p^{a,3}(z))_{21}-(m_p^{a,3}(z))_{22}$$
$$\times\;\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\Big)\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^o_0(z)-\widetilde{\mathcal{R}}^o_0(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)\bigg),\tag{2.106}$$

*and*

$$z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\mathbb{E}\exp\Big(-n(g^o(z)-\ell_o-\mathfrak{Q}^-_{\mathcal{A}})\Big)\bigg((m_p^{a,3}(z))_{12}$$
$$\times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^o_0(z)-\widetilde{\mathcal{R}}^o_0(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)+(m_p^{a,3}(z))_{22}$$
$$\times\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^o_0(z)-\widetilde{\mathcal{R}}^o_0(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)\bigg),\tag{2.107}$$

*where*

$$(m_p^{a,3}(z))_{11}:=-\,\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11}$$
$$-\Big(\omega^2\,\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_j^o}\Big),\tag{2.108}$$

$$(m_p^{a,3}(z))_{12}:=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11}$$
$$\times\,\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho_j^o}-\Big(\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\Big),\tag{2.109}$$

$$(m_p^{a,3}(z))_{21}:=-\,\mathrm{i}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21}$$
$$-\Big(\omega^2\,\mathrm{Ai}(\omega^2p_a)(p_a)^{1/4}+\omega\,\mathrm{Ai}'(\omega^2p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho_j^o}\Big),\tag{2.110}$$

$$(m_p^{a,3}(z))_{22}:=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21}$$
$$\times\,\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho_j^o}-\Big(\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\Big);\tag{2.111}$$

*and* **(12)** *for* $z\in\Omega^{o,4}_{a_j}$ $(\subset\mathbb{C}_-\cap\mathbb{U}^o_{\delta_{a_j}})$, $j=1,\dots,N+1$,

$$z\boldsymbol{\pi}_{2n+1}(z)\underset{n\to\infty}{=}\frac{1}{\mathbb{E}}\exp\Big(n(g^o(z)-\mathfrak{Q}^-_{\mathcal{A}})\Big)\bigg((m_p^{a,4}(z))_{11}\bigg(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^o_0(z)-\widetilde{\mathcal{R}}^o_0(z)\Big)_{11}$$
$$+\;\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\bigg)+(m_p^{a,4}(z))_{21}\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}^o_0(z)-\widetilde{\mathcal{R}}^o_0(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2}^2}\right)\right)\bigg),\tag{2.112}$$

*and*

$$z\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\mathbb{E}\exp\Big(-n(g^o(z)-\ell_o-\mathfrak{Q}^-_{\mathcal{A}})\Big)\Big((m_p^{a,4}(z))_{12}$$

$$\times\left(1+\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{11}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)+(m_p^{a,4}(z))_{22}$$
$$\times\left(\frac{1}{n+\frac{1}{2}}\Big(\mathcal{R}_0^o(z)-\widetilde{\mathcal{R}}_0^o(z)\Big)_{12}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right)\right)\Bigg), \tag{2.113}$$

*where*

$$(m_p^{a,4}(z))_{11} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}-\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11}$$
$$+\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}+\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^o_j}\Big), \tag{2.114}$$

$$(m_p^{a,4}(z))_{12} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{11}$$
$$\times\,\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^o_j}-\Big(\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{12}\Big), \tag{2.115}$$

$$(m_p^{a,4}(z))_{21} := -\mathrm{i}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}-\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21}$$
$$+\Big(\mathrm{Ai}(p_a)(p_a)^{1/4}+\mathrm{Ai}'(p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^o_j}\Big), \tag{2.116}$$

$$(m_p^{a,4}(z))_{22} := \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi^o_{a_j}(z)}\Big(\mathrm{i}\Big(-\omega^2\,\mathrm{Ai}(\omega p_a)(p_b)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{21}$$
$$\times\,\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^o_j}-\Big(\omega^2\,\mathrm{Ai}(\omega p_a)(p_a)^{1/4}+\mathrm{Ai}'(\omega p_a)(p_a)^{-1/4}\Big)(\overset{o}{m}{}^{\infty}(z))_{22}\Big). \tag{2.117}$$

**Remark 2.3.2.** Using limiting values, if necessary, all of the above (asymptotic) formulae for $\boldsymbol{\pi}_{2n+1}(z)$ and $z\int_{\mathbb{R}}(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}(s(s-z))^{-1}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}$ have a natural interpretation on the real and imaginary axes. ■

**Theorem 2.3.2.** *Let all the conditions stated in Theorem* 2.3.1 *be valid, and let* $\overset{o}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ *be the unique solution of* **RHP2**. *Let* $H_k^{(m)}$, $(m,k)\in\mathbb{Z}\times\mathbb{N}$, *be the Hankel determinants associated with the bi-infinite, real-valued, strong moment sequence* $\Big\{c_k=\int_{\mathbb{R}}s^k\mathrm{e}^{-n\widetilde{V}(s)}\,\mathrm{d}s,\ n\in\mathbb{N}\Big\}_{k\in\mathbb{Z}}$ *defined in Equations* (1.1), *and let* $\boldsymbol{\pi}_{2n+1}(z)$ *be the odd degree monic orthogonal L-polynomial defined in Lemma* 2.2.2, *that is,* $z\boldsymbol{\pi}_{2n+1}(z):=(\overset{o}{\mathrm{Y}}(z))_{11}$, *with* $n\to\infty$ *asymptotics (in the entire complex plane) given in Theorem* 2.3.1. *Then,*

$$(\xi^{(2n+1)}_{-n-1})^2=\frac{1}{\|\boldsymbol{\pi}_{2n+1}(\cdot)\|^2_{\mathcal{L}}}=\frac{H^{(-2n)}_{2n+1}}{H^{(-2n-2)}_{2n+1}}\underset{n\to\infty}{=}\frac{\mathrm{e}^{-n\ell_o}}{\pi}\Xi^{\natural}\left(1+\frac{1}{n+\frac{1}{2}}\Xi^{\natural}(\mathfrak{Q}^{\natural})_{12}+\mathcal{O}\left(\frac{c^{\natural}(n)}{(n+\frac{1}{2})^2}\right)\right), \tag{2.118}$$

*where*

$$\Xi^{\natural}:=2\mathbb{E}^2\left(\sum_{k=1}^{N+1}\Big((b^o_{k-1})^{-1}-(a^o_k)^{-1}\Big)\right)^{-1}\frac{\boldsymbol{\theta}^o(\boldsymbol{u}^o_+(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)\boldsymbol{\theta}^o(-\boldsymbol{u}^o_+(0)+\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(-\boldsymbol{u}^o_+(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}^o_+(0)+\boldsymbol{d}_o)}, \tag{2.119}$$

$$\mathfrak{Q}^{\natural}:=-2\mathrm{i}\sum_{j=1}^{N+1}\left(\frac{(\mathcal{A}^o(a^o_j)(\widehat{\alpha}^o_1(a^o_j)+2(a^o_j)^{-1}\widehat{\alpha}^o_0(a^o_j))-\mathcal{B}^o(a^o_j)\widehat{\alpha}^o_0(a^o_j))}{(a^o_j)^2(\widehat{\alpha}^o_0(a^o_j))^2}\right.$$
$$\left.+\frac{(\mathcal{A}^o(b^o_{j-1})(\widehat{\alpha}^o_1(b^o_{j-1})+2(b^o_{j-1})^{-1}\widehat{\alpha}^o_0(b^o_{j-1}))-\mathcal{B}^o(b^o_{j-1})\widehat{\alpha}^o_0(b^o_{j-1}))}{(b^o_{j-1})^2(\widehat{\alpha}^o_0(b^o_{j-1}))^2}\right), \tag{2.120}$$

$(\mathfrak{Q}^{\natural})_{12}$ *denotes the* (1 2)*-element of* $\mathfrak{Q}^{\natural}$, $c^{\natural}(n)=_{n\to\infty}\mathcal{O}(1)$, *and all relevant parameters are defined in Theorem* 2.3.1: *asymptotics for* $\xi^{(2n+1)}_{-n-1}$ *are obtained by taking the positive square root of both sides of Equation* (2.118). *Furthermore, the* $n\to\infty$ *asymptotic expansion (in the entire complex plane) for the odd degree orthonormal L-polynomial,*

$$\phi_{2n+1}(z)=\xi^{(2n+1)}_{-n-1}\boldsymbol{\pi}_{2n+1}(z), \tag{2.121}$$

*to* $\mathcal{O}\Big((n+1/2)^{-2}\Big)$, *is given by the (scalar) multiplication of the* $n\to\infty$ *asymptotics of* $\boldsymbol{\pi}_{2n+1}(z)$ *and* $\xi^{(2n+1)}_{-n-1}$ *stated, respectively, in Theorem* 2.3.1 *and Equations* (2.118)–(2.120).

**Remark 2.3.3.** Since, from general theory (cf. Section 1), and, by construction (cf. Equations (1.3) and (1.8)), $\xi^{(2n+1)}_{-n-1}>0$, it follows, incidentally, from Theorem 2.3.2, Equations (2.118)–(2.120) that: (i) $\Xi^{\natural}>0$; and (ii) $\mathrm{Im}((\mathfrak{Q}^{\natural})_{12})=0$. ■

## 3 The Equilibrium Measure, the Variational Problem, and the Transformed RHP

In this section, the detailed analysis of the 'odd degree' variational problem, and the associated 'odd' equilibrium measure, is undertaken (see Lemmas 3.1–3.3 and Lemma 3.5), including the discussion of the corresponding $g$-function, denoted, herein, as $g^{o}$, and **RHP2**, that is, $(\overset{o}{\mathrm{Y}}(z), \mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_{+}, \mathbb{R})$, is reformulated as an equivalent[6], auxiliary RHP (see Lemma 3.4). The proofs of Lemmas 3.1–3.3 are modelled on the calculations of Saff-Totik ([43], Chapter 1), Deift ([79], Chapter 6), and Johansson [80].

One begins by establishing the existence of the 'odd' equilibrium measure, $\mu_{V}^{o}$ $(\in \mathcal{M}_{1}(\mathbb{R}))$.

**Lemma 3.1.** *Let the external field* $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5), *and set* $w^{o}(z):=\mathrm{e}^{-\widetilde{V}(z)}$. *For* $\mu^{o}\in\mathcal{M}_{1}(\mathbb{R})$, *define the weighted energy functional* $\mathrm{I}_{V}^{o}[\mu^{o}]\colon \mathcal{M}_{1}(\mathbb{R})\to\mathbb{R}$,

$$\mathrm{I}_{V}^{o}[\mu^{o}]:=\iint_{\mathbb{R}^{2}}\ln\Big(|s-t|^{2+\frac{1}{n}}|st|^{-1}w^{o}(s)w^{o}(t)\Big)^{-1}\,\mathrm{d}\mu^{o}(s)\,\mathrm{d}\mu^{o}(t),\qquad n\in\mathbb{N},$$

*and consider the minimisation problem*

$$E_{V}^{o}=\inf\Big\{\mathrm{I}_{V}^{o}[\mu^{o}];\ \mu^{o}\in\mathcal{M}_{1}(\mathbb{R})\Big\}.$$

*Then:* (1) $E_{V}^{o}$ *is finite;* (2) $\exists\ \mu_{V}^{o}\in\mathcal{M}_{1}(\mathbb{R})$ *such that* $\mathrm{I}_{V}^{o}[\mu_{V}^{o}]=E_{V}^{o}$ *(the infimum is attained), and* $\mu_{V}^{o}$ *has finite weighted logarithmic energy* $(-\infty<\mathrm{I}_{V}^{o}[\mu_{V}^{o}]<+\infty)$; *and* (3) $J_{o}:=\operatorname{supp}(\mu_{V}^{o})$ *is compact,* $J_{o}\subset\{z;\ w^{o}(z)>0\}$, *and* $J_{o}$ *has positive logarithmic capacity, that is,* $\operatorname{cap}(J_{o}):=\exp\Big(-\inf\{\mathrm{I}_{V}^{o}[\mu^{o}];\ \mu^{o}\in\mathcal{M}_{1}(J_{o})\}\Big)>0$.

*Proof.* Let $\mu^{o}\in\mathcal{M}_{1}(\mathbb{R})$, and set[7] $w^{o}(z):=\exp(-\widetilde{V}(z))$, where $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies conditions (2.3)–(2.5). From the definition of $\mathrm{I}_{V}^{o}[\mu^{o}]$ given in the Lemma, one shows that, for $n\in\mathbb{N}$,

$$\begin{aligned}\mathrm{I}_{V}^{o}[\mu^{o}]&=\iint_{\mathbb{R}^{2}}\Big(\Big(1+\frac{1}{n}\Big)\ln(|s-t|^{-1})+\ln(|s^{-1}-t^{-1}|^{-1})\Big)\,\mathrm{d}\mu^{o}(s)\,\mathrm{d}\mu^{o}(t)+2\int_{\mathbb{R}}\widetilde{V}(s)\,\mathrm{d}\mu^{o}(s)\\&=:\iint_{\mathbb{R}^{2}}K_{V,n}^{o}(s,t)\,\mathrm{d}\mu^{o}(s)\,\mathrm{d}\mu^{o}(t),\end{aligned}$$

where (the $n$-dependent symmetric kernel)

$$K_{V,n}^{o}(s,t)=K_{V,n}^{o}(t,s):=\Big(1+\frac{1}{n}\Big)\ln(|s-t|^{-1})+\ln(|s^{-1}-t^{-1}|^{-1})+\widetilde{V}(s)+\widetilde{V}(t)$$

(of course, the definition of $\mathrm{I}_{V}^{o}[\mu^{o}]$ only makes sense provided both integrals exist and are finite). Recall the following inequalities (see, for example, Chapter 6 of [79]): $|s-t|\leqslant(1+s^{2})^{1/2}(1+t^{2})^{1/2}$ and $|s^{-1}-t^{-1}|\leqslant(1+s^{-2})^{1/2}(1+t^{-2})^{1/2}$, $s,t\in\mathbb{R}$, whence $\ln(|s-t|^{-1})\geqslant-\frac{1}{2}\ln(1+s^{2})-\frac{1}{2}\ln(1+t^{2})$ and $\ln(|s^{-1}-t^{-1}|^{-1})\geqslant-\frac{1}{2}\ln(1+s^{-2})-\frac{1}{2}\ln(1+t^{-2})$; thus,

$$K_{V,n}^{o}(s,t)\geqslant\frac{1}{2}\Big(2\widetilde{V}(s)-\Big(1+\frac{1}{n}\Big)\ln(s^{2}+1)-\ln(s^{-2}+1)\Big)+\frac{1}{2}\Big(2\widetilde{V}(t)-\Big(1+\frac{1}{n}\Big)\ln(t^{2}+1)-\ln(t^{-2}+1)\Big).$$

Recalling conditions (2.3)–(2.5) for the external field $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$, in particular, $\exists\ \delta_{1}$ $(=\delta_{1}(n))>0$ (resp., $\exists\ \delta_{2}$ $(=\delta_{2}(n))>0$) such that $\widetilde{V}(x)\geqslant(1+\delta_{1})(1+\frac{1}{n})\ln(x^{2}+1)$ (resp., $\widetilde{V}(x)\geqslant(1+\delta_{2})\ln(x^{-2}+1)$) for sufficiently large $|x|$ (resp., small $|x|$), it follows that $2\widetilde{V}(x)-(1+\frac{1}{n})\ln(x^{2}+1)-\ln(x^{-2}+1)\geqslant C_{V}^{o}>-\infty$, whence $K_{V,n}^{o}(s,t)\geqslant C_{V}^{o}$ $(>-\infty)$, which shows that $K_{V,n}^{o}(s,t)$ is bounded from below (on $\mathbb{R}^{2}$); hence,

$$\mathrm{I}_{V}^{o}[\mu^{o}]\geqslant\iint_{\mathbb{R}^{2}}C_{V}^{o}\,\mathrm{d}\mu^{o}(s)\,\mathrm{d}\mu^{o}(t)=C_{V}^{o}\underbrace{\int_{\mathbb{R}}\mathrm{d}\mu^{o}(s)}_{=1}\ \underbrace{\int_{\mathbb{R}}\mathrm{d}\mu^{o}(t)}_{=1}\geqslant C_{V}^{o}\quad(>-\infty).$$

---

[6]If there are two RHPs, $(\mathcal{Y}_{1}(z),v_{1}(z),\Gamma_{1})$ and $(\mathcal{Y}_{2}(z),v_{2}(z),\Gamma_{2})$, say, with $\Gamma_{2}\subset\Gamma_{1}$ and $v_{1}(z)\restriction_{\Gamma_{1}\setminus\Gamma_{2}}=_{n\to\infty}\mathrm{I}+o(1)$, then, within the BC framework [74], and modulo $o(1)$ estimates, their solutions, $\mathcal{Y}_{1}$ and $\mathcal{Y}_{2}$, respectively, are (asymptotically) equal.

[7]All the introduced variables in the proof depend on $n$, which would necessitate the introduction of additional, superfluous notation to encode this $n$ dependence; however, for simplicity of notation, such cumbersome $n$-dependencies will not be introduced, and the reader should be cognizant of this fact: *mutatis mutandis* for the remainder of the paper.

It follows from the above inequality and the definition of $E_V^o$ stated in the Lemma that, $\forall\ \mu^o \in \mathcal{M}_1(\mathbb{R})$, $E_V^o \geqslant C_V^o > -\infty$, which shows that $E_V^o$ is bounded from below. Let $\varepsilon$ be an arbitrarily fixed, sufficiently small positive real number, and set $\Sigma_{o,\varepsilon} := \{z;\ w^o(z) \geqslant \varepsilon\}$; then $\Sigma_{o,\varepsilon}$ is compact, and $\Sigma_{o,0} := \cup_{l=1}^{\infty} \Sigma_{o,1/l} = \cup_{l=1}^{\infty}\{z;\ w^o(z) > l^{-1}\} = \{z;\ w^o(z) > 0\}$. Since, for $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying conditions (2.3)–(2.5), $w^o$ is an *admissible weight* [43], in which case $\Sigma_{o,0}$ has positive logarithmic capacity, that is, $\mathrm{cap}(\Sigma_{o,0}) = \exp(-\inf\{\mathrm{I}_V^o[\mu^o];\ \mu^o \in \mathcal{M}_1(\Sigma_{o,0})\}) > 0$, it follows that $\exists\ l^* \in \mathbb{N}$ such that $\mathrm{cap}(\Sigma_{o,1/l^*}) = \exp(-\inf\{\mathrm{I}_V^o[\mu^o];\ \mu^o \in \mathcal{M}_1(\Sigma_{o,1/l^*})\}) > 0$, which, in turn, means that there exists a probability measure, $\mu_{l^*}^o$, say, with $\mathrm{supp}(\mu_{l^*}^o) \subseteq \Sigma_{o,1/l^*}$, such that $\iint_{\Sigma_{o,1/l^*}^2} \ln(|s-t|^{-(2+\frac{1}{n})}|st|)\,\mathrm{d}\mu_{l^*}^o(s)\,\mathrm{d}\mu_{l^*}^o(t) < +\infty$, where $\Sigma_{o,1/l^*}^2 = \Sigma_{o,1/l^*} \times \Sigma_{o,1/l^*}$ $(\subseteq \mathbb{R}^2)$. For $z \in \mathrm{supp}(\mu_{l^*}^o) \subseteq \Sigma_{o,1/l^*}$, it follows that $w^o(z) \geqslant 1/l^*$, whence $\iint_{\Sigma_{o,1/l^*}^2} \ln(w^o(s)w^o(t))^{-1}\,\mathrm{d}\mu_{l^*}^o(s)\,\mathrm{d}\mu_{l^*}^o(t) \leqslant 2\ln(l^*) < +\infty$ $\Rightarrow$

$$\mathrm{I}_V^o[\mu_{l^*}^o] = \iint_{\Sigma_{o,1/l^*}^2} \ln\Big(|s-t|^{2+\frac{1}{n}}|st|^{-1}w^o(s)w^o(t)\Big)^{-1}\,\mathrm{d}\mu_{l^*}^o(s)\,\mathrm{d}\mu_{l^*}^o(t) < +\infty;$$

thus, it follows that $E_V^o := \inf\{\mathrm{I}_V^o[\mu^o];\ \mu^o \in \mathcal{M}_1(\mathbb{R})\}$ is finite (see, also, below).

Choose a sequence of probability measures $\{\mu_m^o\}_{m=1}^{\infty}$ in $\mathcal{M}_1(\mathbb{R})$ such that $\mathrm{I}_V^o[\mu_m^o] \leqslant E_V^o + \frac{1}{m}$. From the analysis above, it follows that

$$\begin{aligned}\mathrm{I}_V^o[\mu_m^o] &= \iint_{\mathbb{R}^2} K_{V,n}^o(s,t)\,\mathrm{d}\mu_m^o(s)\,\mathrm{d}\mu_m^o(t) \geqslant \iint_{\mathbb{R}^2}\bigg(\frac{1}{2}\Big(2\widetilde{V}(s) - \Big(1+\frac{1}{n}\Big)\ln(s^2+1) - \ln(s^{-2}+1)\Big) \\ &+ \frac{1}{2}\Big(2\widetilde{V}(t) - \Big(1+\frac{1}{n}\Big)\ln(t^2+1) - \ln(t^{-2}+1)\Big)\bigg)\,\mathrm{d}\mu_m^o(s)\,\mathrm{d}\mu_m^o(t).\end{aligned}$$

Set

$$\widehat{\psi}_V^o(z) := 2\widetilde{V}(z) - \Big(1+\frac{1}{n}\Big)\ln(z^2+1) - \ln(z^{-2}+1).$$

Then $\mathrm{I}_V^o[\mu_m^o] \geqslant \int_{\mathbb{R}} \widehat{\psi}_V^o(s)\,\mathrm{d}\mu_m^o(s) \Rightarrow E_V^o + \frac{1}{m} \geqslant \int_{\mathbb{R}} \widehat{\psi}_V^o(s)\,\mathrm{d}\mu_m^o(s)$. Recalling that $\exists\ \delta_1 > 0$ (resp., $\exists\ \delta_2 > 0$) such that $\widetilde{V}(x) \geqslant (1+\delta_1)(1+\frac{1}{n})\ln(x^2+1)$ (resp., $\widetilde{V}(x) \geqslant (1+\delta_2)\ln(x^{-2}+1)$) for sufficiently large $|x|$ (resp., small $|x|$), it follows that, for any $b_o > 0$, $\exists\ M_o > 1$ such that $\widehat{\psi}_V^o(z) > b_o\ \forall\ z \in \{|z| \geqslant M_o\} \cup \{|z| \leqslant M_o^{-1}\} =: \mathfrak{D}_o$, which implies that

$$\begin{aligned}E_V^o + \frac{1}{m} &\geqslant \int_{\mathbb{R}} \widehat{\psi}_V^o(s)\,\mathrm{d}\mu_m^o(s) = \int_{\mathfrak{D}_o} \underbrace{\widehat{\psi}_V^o(s)}_{>b_o}\,\mathrm{d}\mu_m^o(s) + \int_{\mathbb{R}\setminus\mathfrak{D}_o} \underbrace{\widehat{\psi}_V^o(s)}_{\geqslant -|C_V^o|}\,\mathrm{d}\mu_m^o(s) \\ &\geqslant b_o \int_{\mathfrak{D}_o} \mathrm{d}\mu_m^o(s) - |C_V^o| \underbrace{\int_{\mathbb{R}\setminus\mathfrak{D}_o} \mathrm{d}\mu_m^o(s)}_{\in[0,1]} \geqslant b_o \int_{\mathfrak{D}_o} \mathrm{d}\mu_m^o(s) - |C_V^o|;\end{aligned}$$

thus,

$$\int_{\mathfrak{D}_o} \mathrm{d}\mu_m^o(s) \leqslant b_o^{-1}\Big(E_V^o + |C_V^o| + \frac{1}{m}\Big),$$

whence

$$\limsup_{m\to\infty} \int_{\mathfrak{D}_o} \mathrm{d}\mu_m^o(s) \leqslant \limsup_{m\to\infty}\Big(b_o^{-1}\Big(E_V^o + |C_V^o| + \frac{1}{m}\Big)\Big).$$

By the Archimedean property, it follows that, $\forall\ \epsilon_o > 0$, $\exists\ N \in \mathbb{N}$ such that, $\forall\ m > N \Rightarrow m^{-1} < \epsilon_o$; thus, choosing $b_o = \epsilon^{-1}(E_V^o + |C_V^o| + \epsilon_o)$, where $\epsilon$ is some arbitrarily fixed, sufficiently small positive real number, it follows that $\limsup_{m\to\infty} \int_{\mathfrak{D}_o} \mathrm{d}\mu_m^o(s) \leqslant \epsilon \Rightarrow$ the sequence of probability measures $\{\mu_m^o\}_{m=1}^{\infty}$ in $\mathcal{M}_1(\mathbb{R})$ is *tight* [80] (that is, given $\epsilon > 0$, $\exists\ M > 1$ such that $\limsup_{m\to\infty} \mu_m^o(\{|s| \geqslant M\} \cup \{|s| \leqslant M^{-1}\}) := \limsup_{m\to\infty} \int_{\{|s|\geqslant M\}\cup\{|s|\leqslant M^{-1}\}} \mathrm{d}\mu_m^o(s) \leqslant \epsilon$). Since the sequence of probabilty measures $\{\mu_m^o\}_{m=1}^{\infty}$ in $\mathcal{M}_1(\mathbb{R})$ is tight, by a Helly Selection Theorem, there exists a (weak$^*$ convergent) subsequence of probability measures $\{\mu_{m_k}^o\}_{k=1}^{\infty}$ in $\mathcal{M}_1(\mathbb{R})$ converging (weakly) to a probability measure $\mu^o \in \mathcal{M}_1(\mathbb{R})$, symbolically $\mu_{m_k}^o \overset{*}{\to} \mu^o$ as $k \to \infty$[8]. One now shows that, if $\mu_m^o \overset{*}{\to} \mu^o$, $\mu_m^o, \mu^o \in \mathcal{M}_1(\mathbb{R})$, then $\liminf_{m\to\infty} \mathrm{I}_V^o[\mu_m^o] \geqslant \mathrm{I}_V^o[\mu^o]$.

[8] A sequence of probability measures $\{\mu_m\}_{m=1}^{\infty}$ in $\mathcal{M}_1(D)$ is said to *converge weakly* as $m \to \infty$ to $\mu \in \mathcal{M}_1(D)$, symbolically $\mu_m \overset{*}{\to} \mu$, if $\mu_m(f) := \int_D f(s)\,\mathrm{d}\mu_m(s) \to \int_D f(s)\,\mathrm{d}\mu(s) =: \mu(f)$ as $m \to \infty\ \forall\ f \in \mathrm{C}_{\mathbf{b}}^{\mathbf{0}}(D)$, where $\mathrm{C}_{\mathbf{b}}^{\mathbf{0}}(D)$ denotes the set of all bounded, continuous functions on $D$ with compact support.

Since $w^o$ is continuous, thus upper semi-continuous [43], there exists a sequence $\{w_m^o\}_{m=1}^{\infty}$ (resp., $\{\widetilde{V}_m\}_{m=1}^{\infty}$) of continuous functions on $\mathbb{R}$ such that $w_{m+1}^o \leqslant w_m^o$ (resp., $\widetilde{V}_{m+1} \geqslant \widetilde{V}_m$)[9], $m \in \mathbb{N}$, and $w_m^o(z) \searrow w^o(z)$ (resp., $\widetilde{V}_m(z) \nearrow \widetilde{V}(z)$) as $m \to \infty$ for every $z \in \mathbb{R}$; in particular,

$$\mathrm{I}_V^o[\mu_k^o] = \iint_{\mathbb{R}^2} K_{V,n}^o(s,t)\,\mathrm{d}\mu_k^o(s)\,\mathrm{d}\mu_k^o(t) \geqslant \iint_{\mathbb{R}^2} K_{V_m,n}^o(s,t)\,\mathrm{d}\mu_k^o(s)\,\mathrm{d}\mu_k^o(t).$$

For arbitrary $q \in \mathbb{R}$, $\mathrm{I}_V^o[\mu_k^o] \geqslant \iint_{\mathbb{R}^2} p^o(s,t;n)\,\mathrm{d}\mu_k^o(s)\,\mathrm{d}\mu_k^o(t)$, where $p^o(s,t;n) := \min\left\{q, K_{V_m,n}^o(s,t)\right\} = p^o(t,s;n)$ (bounded and continuous on $\mathbb{R}^2$). Recall that $\{\mu_m^o\}_{m=1}^{\infty}$ is tight in $\mathcal{M}_1(\mathbb{R})$. For $M_o > 1$, let $h_M^o(x) \in \mathrm{C}_{\mathbf{b}}^{\mathbf{0}}(\mathbb{R})$ be such that:

(i) $h_M^o(x) = 1$, $x \in [-M_o, -M_o^{-1}] \cup [M_o^{-1}, M_o] =: \mathfrak{D}_{M_o}$;
(ii) $h_M^o(x) = 0$, $x \in \mathbb{R} \setminus \mathfrak{D}_{M_o+1}$; and
(iii) $0 \leqslant h_M^o(x) \leqslant 1$, $x \in \mathbb{R}$.

Note the decomposition $\iint_{\mathbb{R}^2} p^o(t,s;n)\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s) = I_a + I_b + I_c$, where

$$\begin{aligned}
I_a &:= \iint_{\mathbb{R}^2} p^o(t,s;n)(1 - h_M^o(s))\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s),\\
I_b &:= \iint_{\mathbb{R}^2} p^o(t,s;n)h_M^o(s)(1 - h_M^o(t))\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s),\\
I_c &:= \iint_{\mathbb{R}^2} p^o(t,s;n)h_M^o(t)h_M^o(s)\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s).
\end{aligned}$$

One shows that, for $n \in \mathbb{N}$,

$$\begin{aligned}
|I_a| &\leqslant \iint_{\mathbb{R}^2} |p^o(t,s;n)|(1 - h_M^o(s))\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s)\\
&\leqslant \sup_{(t,s)\in\mathbb{R}^2} |p^o(t,s;n)| \underbrace{\int_{\mathbb{R}} \mathrm{d}\mu_k^o(t)}_{=1} \left( \int_{\mathfrak{D}_{M_o}} \underbrace{(1 - h_M^o(s))}_{=0}\,\mathrm{d}\mu_k^o(s) + \int_{\mathbb{R}\setminus\mathfrak{D}_{M_o+1}} (1 - \underbrace{h_M^o(s))}_{=0}\,\mathrm{d}\mu_k^o(s) \right),
\end{aligned}$$

whence

$$\limsup_{k\to\infty} |I_a| \leqslant \sup_{(t,s)\in\mathbb{R}^2} |p^o(t,s;n)| \underbrace{\limsup_{k\to\infty} \int_{\mathbb{R}\setminus\mathfrak{D}_{M_o+1}} \mathrm{d}\mu_k^o(s)}_{\leqslant\epsilon} \leqslant \epsilon \sup_{(t,s)\in\mathbb{R}^2} |p^o(t,s;n)|;$$

similarly,

$$\limsup_{k\to\infty} |I_b| \leqslant \epsilon \sup_{(t,s)\in\mathbb{R}^2} |p^o(t,s;n)|.$$

Since, for $n \in \mathbb{N}$, $p^o(t,s;n)$ is continuous and bounded on $\mathbb{R}^2$, there exists, by a generalisation of the Stone-Weierstrass Theorem (for the single-variable case), a polynomial in two variables (with $n$-dependent coefficients), $p(t,s;n)$, say, with $p(t,s;n) = \sum_{i\geqslant i_o} \sum_{j\geqslant j_o} \gamma_{ij}(n) t^i s^j$, such that $|p^o(t,s;n) - p(t,s;n)| \leqslant \epsilon(n) := \epsilon$; thus,

$$|h_M^o(t)h_M^o(s)p^o(t,s;n) - h_M^o(t)h_M^o(s)p(t,s;n)| \leqslant \epsilon, \quad t,s \in \mathbb{R}.$$

Rewrite $I_c$ as

$$\begin{aligned}
I_c &= \iint_{\mathbb{R}^2} h_M^o(s)h_M^o(t)p(t,s;n)\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s) + \iint_{\mathbb{R}^2} h_M^o(s)h_M^o(t)(p^o(t,s;n) - p(t,s;n))\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s)\\
&=: I_c^{\alpha} + I_c^{\beta}.
\end{aligned}$$

One now shows that

$$|I_c^{\beta}| \leqslant \iint_{\mathbb{R}^2} h_M^o(s)h_M^o(t)\,\underbrace{|p^o(t,s;n) - p(t,s;n)|}_{\leqslant\epsilon}\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s) \leqslant \epsilon \int_{\mathbb{R}} h_M^o(s)\,\mathrm{d}\mu_k^o(s) \int_{\mathbb{R}} h_M^o(t)\,\mathrm{d}\mu_k^o(t)$$

[9]Adding a suitable constant, if necessary, which does not change $\mu_m^o$, or the regularity of $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, one may assume that $\widetilde{V} \geqslant 0$ and $\widetilde{V}_m \geqslant 0$, $m \in \mathbb{N}$.

$$\leqslant \epsilon \left( \int_{\mathfrak{D}_{M_o}} \underbrace{h_M^o(s)}_{=1} \, \mathrm{d}\mu_k^o(s) + \int_{\mathbb{R} \setminus \mathfrak{D}_{M_o+1}} \underbrace{h_M^o(s)}_{=0} \, \mathrm{d}\mu_k^o(s) \right)^2 \leqslant \epsilon \left( \int_{\mathfrak{D}_{M_o}} \mathrm{d}\mu_k^o(s) \right)^2$$

$$\leqslant \epsilon \left( \int_{\mathbb{R}} \mathrm{d}\mu_k^o(s) \right)^2 \leqslant \epsilon,$$

and

$$\begin{aligned}
I_c^\alpha &= \iint_{\mathbb{R}^2} h_M^o(s) h_M^o(t) \sum_{i \geqslant i_o} \sum_{j \geqslant j_o} \gamma_{ij}(n) t^i s^j \, \mathrm{d}\mu_k^o(t) \, \mathrm{d}\mu_k^o(s) \\
&= \sum_{i \geqslant i_o} \sum_{j \geqslant j_o} \gamma_{ij}(n) \left( \int_{\mathbb{R}} h_M^o(t) t^i \, \mathrm{d}\mu_k^o(t) \right) \left( \int_{\mathbb{R}} h_M^o(s) s^j \, \mathrm{d}\mu_k^o(s) \right) \\
&\to \sum_{i \geqslant i_o} \sum_{j \geqslant j_o} \gamma_{ij}(n) \left( \int_{\mathbb{R}} h_M^o(t) t^i \, \mathrm{d}\mu^o(t) \right) \left( \int_{\mathbb{R}} h_M^o(s) s^j \, \mathrm{d}\mu^o(s) \right) \qquad (\text{since } \mu_k^o \overset{*}{\to} \mu^o \text{ as } k \to \infty) \\
&= \iint_{\mathbb{R}^2} \left( \sum_{i \geqslant i_o} \sum_{j \geqslant j_o} \gamma_{ij}(n) t^i s^j \right) h_M^o(t) h_M^o(s) \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s),
\end{aligned}$$

whence, recalling that $p(t,s;n) = \sum_{i \geqslant i_o} \sum_{j \geqslant j_o} \gamma_{ij}(n) t^i s^j$, it follows that

$$I_c^\alpha = \iint_{\mathbb{R}^2} p(t,s;n) h_M^o(t) h_M^o(s) \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s).$$

Furthermore, for $n \in \mathbb{N}$,

$$I_c^\alpha \leqslant \iint_{\mathbb{R}^2} p^o(t,s;n) h_M^o(t) h_M^o(s) \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) + \epsilon \underbrace{\int_{\mathbb{R}} h_M^o(t) \, \mathrm{d}\mu^o(t)}_{\leqslant 1} \underbrace{\int_{\mathbb{R}} h_M^o(s) \, \mathrm{d}\mu^o(s)}_{\leqslant 1} \Rightarrow$$

$$\begin{aligned}
I_c^\alpha &\leqslant \iint_{\mathbb{R}^2} p^o(t,s;n) |1 + (h_M^o(t) - 1)| |1 + (h_M^o(s) - 1)| \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) + \epsilon \\
&\leqslant \iint_{\mathbb{R}^2} p^o(t,s;n) \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) + \iint_{\mathbb{R}^2} p^o(t,s;n) |h_M^o(s) - 1| \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) + \epsilon \\
&+ \iint_{\mathbb{R}^2} p^o(t,s;n) |h_M^o(t) - 1| \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) + \iint_{\mathbb{R}^2} p^o(t,s;n) |h_M^o(t) - 1| |h_M^o(s) - 1| \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) \\
&\leqslant \iint_{\mathbb{R}^2} p^o(t,s;n) \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) + 2 \sup_{(t,s) \in \mathbb{R}^2} |p^o(t,s;n)| \underbrace{\int_{(\mathbb{R} \setminus \mathfrak{D}_{M_o}) \cup \mathfrak{D}_{M_o}} |h_M^o(s) - 1| \, \mathrm{d}\mu^o(s)}_{\leqslant \epsilon} \underbrace{\int_{\mathbb{R}} \mathrm{d}\mu^o(t)}_{=1} \\
&+ \sup_{(t,s) \in \mathbb{R}^2} |p^o(t,s;n)| \left( \underbrace{\int_{(\mathbb{R} \setminus \mathfrak{D}_{M_o}) \cup \mathfrak{D}_{M_o}} |h_M^o(t) - 1| \, \mathrm{d}\mu^o(t)}_{\leqslant \epsilon} \right)^2 + \epsilon \\
&\leqslant \iint_{\mathbb{R}^2} p^o(t,s;n) \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) + \epsilon \left( 1 + 2 \sup_{(t,s) \in \mathbb{R}^2} |p^o(t,s;n)| \right) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

whereupon, neglecting the $\mathcal{O}(\epsilon^2)$ term, and setting $\varkappa_n^\natural := 1 + 2 \sup_{(t,s) \in \mathbb{R}^2} |p^o(t,s;n)|$, one obtains

$$I_c^\alpha \leqslant \iint_{\mathbb{R}^2} p^o(t,s;n) \, \mathrm{d}\mu^o(t) \, \mathrm{d}\mu^o(s) + \varkappa_n^\natural \epsilon.$$

Hence, assembling the above-derived bounds for $I_a$, $I_b$, $I_c^{\beta}$, and $I_c^{\alpha}$, one arrives at, for $n \in \mathbb{N}$, upon setting $\epsilon_n^{\natural} := 2\varkappa_n^{\natural}\epsilon$,

$$\iint_{\mathbb{R}^2} p^o(t,s;n)\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s) - \iint_{\mathbb{R}^2} p^o(t,s;n)\,\mathrm{d}\mu^o(t)\,\mathrm{d}\mu^o(s) \leqslant \epsilon_n^{\natural};$$

thus,

$$\iint_{\mathbb{R}^2} p^o(t,s;n)\,\mathrm{d}\mu_k^o(t)\,\mathrm{d}\mu_k^o(s) \to \iint_{\mathbb{R}^2} p^o(t,s;n)\,\mathrm{d}\mu^o(t)\,\mathrm{d}\mu^o(s) \quad \text{as } k \to \infty.$$

Recalling that $p^o(t,s;n) := \min\left\{q, K_{V_m,n}^o(t,s)\right\}$, $(q,m) \in \mathbb{R} \times \mathbb{N}$, it follows from the above analysis that, for $n \in \mathbb{N}$,

$$\liminf_{k\to\infty} \mathrm{I}_V^o[\mu_k^o] \geqslant \iint_{\mathbb{R}^2} \min\left\{q, K_{V_m,n}^o(t,s)\right\}\mathrm{d}\mu^o(t)\,\mathrm{d}\mu^o(s):$$

letting $q \uparrow \infty$ and $m \to \infty$, and using the Monotone Convergence Theorem, one arrives at, upon noting that $\min\left\{q, K_{V_m,n}^o(t,s)\right\} \to K_{V,n}^o(t,s)$,

$$\liminf_{k\to\infty} \mathrm{I}_V^o[\mu_k^o] \geqslant \iint_{\mathbb{R}^2} K_{V,n}^o(t,s)\,\mathrm{d}\mu^o(t)\,\mathrm{d}\mu^o(s) = \mathrm{I}_V^o[\mu^o], \quad \mu_k^o, \mu^o \in \mathcal{M}_1(\mathbb{R}).$$

Since, from the analysis above, it was shown that there exists a weakly (weak$^*$) convergent subsequence (of probability measures) $\{\mu_{m_k}^o\}_{k=1}^{\infty}$ $(\subset \mathcal{M}_1(\mathbb{R}))$ of $\{\mu_m^o\}_{m=1}^{\infty}$ $(\subset \mathcal{M}_1(\mathbb{R}))$ with a weak limit $\mu^o \in \mathcal{M}_1(\mathbb{R})$, namely, $\mu_{m_k}^o \to \mu^o$ as $k \to \infty$, upon recalling that $\mathrm{I}_V^o[\mu_m^o] \leqslant E_V^o + \frac{1}{m}$, $m \in \mathbb{N}$, it follows that, in the limit as $m \to \infty$, $\mathrm{I}_V^o[\mu^o] \leqslant E_V^o := \inf\{\mathrm{I}_V^o[\mu^o];\, \mu^o \in \mathcal{M}_1(\mathbb{R})\}$; from the latter two inequalities, it follows, thus, that $\exists\ \mu^o := \mu_V^o \in \mathcal{M}_1(\mathbb{R})$, the 'odd' equilibrium measure, such that $\mathrm{I}_V^o[\mu_V^o] = \inf\{\mathrm{I}_V^o[\mu^o];\, \mu^o \in \mathcal{M}_1(\mathbb{R})\}$, that is, the infimum is attained (the uniqueness of $\mu_V^o \in \mathcal{M}_1(\mathbb{R})$ is proven in Lemma 3.3 below).

The compactness of $\mathrm{supp}(\mu_V^o) =: J_o$ is now established: actually, the following proof is true for any $\mu \in \mathcal{M}_1(\mathbb{R})$ achieving the above minimum; in particular, for $\mu = \mu_V^o$. Without loss of generality, therefore, let $\mu_w \in \mathcal{M}_1(\mathbb{R})$ be such that $\mathrm{I}_V^o[\mu_w] = E_V^o$, and let $D$ be any proper subset of $\mathbb{R}$ for which $\mu_w(D) := \int_D \mathrm{d}\mu_w(s) > 0$. As in [80], set

$$\mu_w^{\varepsilon}(z) := (1+\varepsilon\mu_w(D))^{-1}\left(\mu_w(z) + \varepsilon(\mu_w \restriction_D)(z)\right), \quad \varepsilon \in (-1,1),$$

where $\mu_w \restriction_D$ denotes the restriction of $\mu_w$ to $D$ (note, also, that $\mu_w^{\varepsilon} > 0$ and bounded, and $\int_{\mathbb{R}} \mathrm{d}\mu_w^{\epsilon}(s) = 1$). Using the fact that $K_{V,n}^o(s,t) = K_{V,n}^o(t,s)$, one shows that, for $n \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{I}_V^o[\mu_w^{\varepsilon}] &= \iint_{\mathbb{R}^2} K_{V,n}^o(s,t)\,\mathrm{d}\mu_w^{\varepsilon}(s)\,\mathrm{d}\mu_w^{\varepsilon}(t) \\
&= (1+\varepsilon\mu_w(D))^{-2} \iint_{\mathbb{R}^2} K_{V,n}^o(s,t)(\mathrm{d}\mu_w(s) + \varepsilon\mathrm{d}(\mu_w\restriction_D)(s))(\mathrm{d}\mu_w(t) + \varepsilon\mathrm{d}(\mu_w\restriction_D)(t)) \\
&= (1+\varepsilon\mu_w(D))^{-2}\Bigg(\mathrm{I}_V^o[\mu_w] + 2\varepsilon \iint_{\mathbb{R}^2} K_{V,n}^o(s,t)\,\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t) \\
&\quad + \varepsilon^2 \iint_{\mathbb{R}^2} K_{V,n}^o(s,t)\,\mathrm{d}(\mu_w\restriction_D)(t)\,\mathrm{d}(\mu_w\restriction_D)(s)\Bigg).
\end{aligned}$$

(Note that all the above integrals are finite due to the argument at the beginning of the proof.) By the minimal property of $\mu_w \in \mathcal{M}_1(\mathbb{R})$, it follows that $\partial_{\varepsilon}\mathrm{I}_V^o[\mu_w^{\varepsilon}] = 0$, which implies that, for $n \in \mathbb{N}$,

$$\iint_{\mathbb{R}^2} (K_{V,n}^o(s,t) - \mathrm{I}_V^o[\mu_w])\,\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t) = 0;$$

but, recalling that, for $\widehat{\psi}_V^o(z) := 2\widetilde{V}(z) - (1+\frac{1}{n})\ln(z^2+1) - \ln(z^{-2}+1)$, $K_{V,n}^o(t,s) \geqslant \frac{1}{2}\psi_V^o(s) + \frac{1}{2}\psi_V^o(t)$, it follows from the above that,

$$\iint_{\mathbb{R}^2} \mathrm{I}_V^o[\mu_w]\,\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t) \geqslant \iint_{\mathbb{R}^2} \left(\tfrac{1}{2}\widehat{\psi}_V^o(s) + \tfrac{1}{2}\widehat{\psi}_V^o(t)\right)\mathrm{d}\mu_w(s)\,\mathrm{d}(\mu_w\restriction_D)(t) \Rightarrow$$

$$0 \geqslant \iint_{\mathbb{R}^2} \left(\tfrac{1}{2}\widehat{\psi}_V^o(s) + \tfrac{1}{2}\widehat{\psi}_V^o(t) - \mathrm{I}_V^o[\mu_w]\right) \mathrm{d}\mu_w(s)\, \mathrm{d}(\mu_w \restriction_D)(t),$$

whence

$$\int_{\mathbb{R}} \left(\widehat{\psi}_V^o(t) + \left(\int_{\mathbb{R}} \widehat{\psi}_V^o(s)\, \mathrm{d}\mu_w(s)\right) - 2\mathrm{I}_V^o[\mu_w]\right) \mathrm{d}(\mu_w \restriction_D)(t) \leqslant 0.$$

Recalling that

$$\widehat{\psi}_V^o(x) := 2\widetilde{V}(x) - \left(1 + \frac{1}{n}\right) \ln(x^2+1) - \ln(x^{-2}+1) = \begin{cases} +\infty, & |x| \to \infty, \\ +\infty, & |x| \to 0, \end{cases}$$

it follows that, $\exists\, T_m := T_m(n) > 1$ such that

$$\widehat{\psi}_V^o(t) + \int_{\mathbb{R}} \widehat{\psi}_V^o(s)\, \mathrm{d}\mu_w(s) - 2\mathrm{I}_V^o[\mu_w] \geqslant 1 \quad \text{for} \quad t \in \left((-T_m, -T_m^{-1}) \cup (T_m^{-1}, T_m)\right)^c$$

(note, also, that $+\infty > \mathrm{I}_V^o[\mu_w] = \iint_{\mathbb{R}^2} K_{V,n}^o(t,s)\, \mathrm{d}\mu_w(t)\, \mathrm{d}\mu_w(s) = \int_{\mathbb{R}} \widehat{\psi}_V^o(\xi)\, \mathrm{d}\mu_w(\xi) =$ a finite real number). Hence, if $D\ (\subset \mathbb{R})$ is such that $D \subset (\{|x| \geqslant T_m\} \cup \{|x| \leqslant T_m^{-1}\})$, $T_m > 1$, it follows from the above calculations that, for $n \in \mathbb{N}$,

$$0 \geqslant \int_{\mathbb{R}} \left(\widehat{\psi}_V^o(t) + \left(\int_{\mathbb{R}} \widehat{\psi}_V^o(s)\, \mathrm{d}\mu_w(s)\right) - 2\mathrm{I}_V^o[\mu_w]\right) \mathrm{d}(\mu_w \restriction_D)(t) \geqslant 1,$$

which is a contradiction; hence, $\operatorname{supp}(\mu_w) \subseteq [-T_m, -T_m^{-1}] \cup [T_m^{-1}, T_m]$, $T_m > 1$; in particular, $J_o := \operatorname{supp}(\mu_V^o) \subseteq [-T_m, -T_m^{-1}] \cup [T_m^{-1}, T_m]$, $T_m > 1$, which establishes the compactness of the support of the 'odd' equilibrium measure $\mu_V^o \in \mathcal{M}_1(\mathbb{R})$. Furthermore, it is worth noting that, since $J_o := \operatorname{supp}(\mu_V^o) =$ compact $(\subset \overline{\mathbb{R}} \setminus \{0, \pm\infty\})$, and $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is real analytic on $J_o$, for $n \in \mathbb{N}$,

$$\begin{aligned} +\infty > E_V^o\ (= \mathrm{I}_V^o[\mu_V^o]) &\geqslant \iint_{\mathbb{R}^2} \ln\left(|s-t|^{2+\frac{1}{n}} |st|^{-1} w^o(s) w^o(t)\right)^{-1} \mathrm{d}\mu_V^o(s)\, \mathrm{d}\mu_V^o(t) \\ &= \iint_{J_o^2} \ln\left(|s-t|^{2+\frac{1}{n}} |st|^{-1} w^o(s) w^o(t)\right)^{-1} \mathrm{d}\mu_V^o(s)\, \mathrm{d}\mu_V^o(t) > -\infty; \end{aligned}$$

moreover, a straightforward consequence of the fact just established is that $J_o$ has positive logarithmic capacity, that is, $\operatorname{cap}(J_o) = \exp(-E_V^o) > 0$. □

**Remark 3.1.** It is important to note from the latter part of the proof of Lemma 3.1 that $J_o \not\supseteq \{0, \pm\infty\}$. This can also be seen as follows. For $\varepsilon$ some arbitrarily fixed, sufficiently small positive real number and $\Sigma_\varepsilon := \{z;\, w^o(z) \geqslant \varepsilon\}$, if $(s,t) \notin \Sigma_\varepsilon \times \Sigma_\varepsilon$, then, for $n \in \mathbb{N}$, $\ln(|s-t|^{2+\frac{1}{n}} |st|^{-1} w^o(s) w^o(t))^{-1} =: K_{V,n}^o(s,t)$ $(= K_{V,n}^o(t,s)) > E_V^o + 1$, which is a contradiction, since it was established above that the minimum is attained $\Leftrightarrow (s,t) \in \Sigma_\varepsilon \times \Sigma_\varepsilon$. Towards this end, it is enough to show that (see, for example, [43]), if $\{(s_m, t_m)\}_{m=1}^\infty$ is a sequence with $\lim \min_{m\to\infty}\{w^o(s_m), w^o(t_m)\} = 0$, then, for $n \in \mathbb{N}$, $\lim_{m\to\infty} \ln(|s_m - t_m|^{2+\frac{1}{n}} |s_m t_m|^{-1} w^o(s_m) w^o(t_m))^{-1} = \lim_{m\to\infty} K_{V,n}^o(s_m, t_m) = +\infty$. Without loss of generality, one can assume that $s_m \to s$ and $t_m \to t$ as $m \to \infty$, where $s$, $t$, or both may be infinite; thus, there are several cases to consider:

(i) if $s$ and $t$ are finite, then, from $\lim \min_{m\to\infty}\{w^o(s_m), w^o(t_m)\} = \min\{w^o(s), w^o(t)\} = 0$, it is clear that $\lim_{m\to\infty} K_{V,n}^o(s_m, t_m) = +\infty$;

(ii) if $|s| = \infty$ (resp., $|t| = \infty$) but $t =$ finite (resp., $s =$ finite), then, due to the fact that $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfies (for $n \in \mathbb{N}$) the conditions

$$2\widetilde{V}(x) - \left(1 + \frac{1}{n}\right) \ln(x^2+1) - \ln(x^{-2}+1) = \begin{cases} +\infty, & |x| \to \infty, \\ +\infty, & |x| \to 0, \end{cases}$$

it follows that $\lim_{m\to\infty} K_{V,n}^o(s_m, t_m) = +\infty$;

(iii) if $|s| = 0$ (resp., $|t| = 0$) but $t =$ finite (resp., $s =$ finite), then, as a result of the above conditions for $\widetilde{V}$, it follows that $\lim_{m\to\infty} K_{V,n}^o(s_m, t_m) = +\infty$;

(iv) if $|s| = \infty$ and $|t| = \infty$, then, again due to the above conditions for $\widetilde{V}$, it follows that $\lim_{m\to\infty} K_{V,n}^o$ $(s_m, t_m) = +\infty$; and

(v) if $|s|=0$ and $|t|=0$, then, again, as above, it follows that $\lim_{m\to\infty} K^{o}_{V,n}(s_m,t_m)=+\infty$.
Hence, for $n\in\mathbb{N}$, $K^{o}_{V,n}(s,t)>E^{o}_{V}+1$ if $(s,t)\notin\Sigma_{\varepsilon}\times\Sigma_{\varepsilon}$, that is, if $s$, $t$, or both $\in\{0,\pm\infty\}$ (which can not be the case, as the infimum $E^{o}_{V}$ is attained $\Leftrightarrow (s,t)\in\Sigma_{\varepsilon}\times\Sigma_{\varepsilon}$, whence $\operatorname{supp}(\mu^{o}_{V})=: J_o \not\supseteq \{0,\pm\infty\}$). ∎

In order to demonstrate the uniqueness of the 'odd' equilibrium measure, $\mu^{o}_{V}$ $(\in\mathcal{M}_1(\mathbb{R}))$, the following lemma is requisite.

**Lemma 3.2.** *Let $\mu:=\mu_1-\mu_2$, where $\mu_1,\mu_2$ are non-negative, finite-moment ($\int_{\operatorname{supp}(\mu_j)} s^m\,\mathrm{d}\mu_j(s)<\infty$, $m\in\mathbb{Z}$, $j=1,2$) measures on $\mathbb{R}$ supported on distinct sets ($\operatorname{supp}(\mu_1)\cap\operatorname{supp}(\mu_2)=\varnothing$), be the (unique) Jordan decomposition of the finite-moment signed measure on $\mathbb{R}$ with mean zero, that is, $\int_{\operatorname{supp}(\mu)}\mathrm{d}\mu(s)=0$, and with $\operatorname{supp}(\mu)=$ compact. Suppose that $-\infty<\iint_{\mathbb{R}^2}\ln(|s-t|^{-(2+\frac{1}{n})}|st|)\,\mathrm{d}\mu_j(s)\,\mathrm{d}\mu_j(t)<+\infty$, $n\in\mathbb{N}$, $j=1,2$. Then, for $n\in\mathbb{N}$,*

$$\iint_{\mathbb{R}^2}\ln\left(\frac{|st|}{|s-t|^{2+\frac{1}{n}}}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=\iint_{\mathbb{R}^2}\ln\left(\frac{|s-t|^{2+\frac{1}{n}}}{|st|}w^{o}(s)w^{o}(t)\right)^{-1}\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\geqslant 0,$$

*where equality holds if, and only if, $\mu=0$.*

*Proof.* Recall the following identity [79][10] (see pg. 147, Equation (6.44)): for $\xi\in\mathbb{R}$ and any $\varepsilon>0$,

$$\ln(\xi^2+\varepsilon^2)=\ln(\varepsilon^2)+2\,\mathrm{Im}\left(\int_0^{+\infty}\left(\frac{\mathrm{e}^{\mathrm{i}\xi v}-1}{\mathrm{i}v}\right)\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right);$$

thus, it follows that, for $n\in\mathbb{N}$,

$$\begin{aligned}\left(1+\frac{1}{2n}\right)\iint_{\mathbb{R}^2}\ln((s-t)^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)&=\left(1+\frac{1}{2n}\right)\iint_{\mathbb{R}^2}\ln(\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\&\quad+2\left(1+\frac{1}{2n}\right)\iint_{\mathbb{R}^2}\left(\mathrm{Im}\left(\int_0^{+\infty}\left(\frac{\mathrm{e}^{\mathrm{i}(s-t)v}-1}{\mathrm{i}v}\right)\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right)\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t),\\
\iint_{\mathbb{R}^2}\ln(s^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)&=\iint_{\mathbb{R}^2}\ln(\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\&\quad+\iint_{\mathbb{R}^2}\left(2\,\mathrm{Im}\left(\int_0^{+\infty}\left(\frac{\mathrm{e}^{\mathrm{i}sv}-1}{\mathrm{i}v}\right)\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right)\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t),\\
\iint_{\mathbb{R}^2}\ln(t^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)&=\iint_{\mathbb{R}^2}\ln(\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\&\quad+\iint_{\mathbb{R}^2}\left(2\,\mathrm{Im}\left(\int_0^{+\infty}\left(\frac{\mathrm{e}^{\mathrm{i}tv}-1}{\mathrm{i}v}\right)\mathrm{e}^{-\varepsilon v}\,\mathrm{d}v\right)\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t);\end{aligned}$$

but, since $\iint_{\mathbb{R}^2}\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=\left(\int_{\mathbb{R}}\mathrm{d}\mu(s)\right)^2=0$, one obtains, after some rearrangement,

$$\begin{aligned}\left(1+\frac{1}{2n}\right)\iint_{\mathbb{R}^2}\ln((s-t)^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)&=2\left(1+\frac{1}{2n}\right)\mathrm{Im}\left(\int_0^{+\infty}\mathrm{e}^{-\varepsilon v}\left(\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}(s-t)v}-1}{\mathrm{i}v}\right)\right.\right.\\&\qquad\left.\left.\times\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\right)\mathrm{d}v\right),\\
\iint_{\mathbb{R}^2}\ln(s^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)&=2\,\mathrm{Im}\left(\int_0^{+\infty}\mathrm{e}^{-\varepsilon v}\left(\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}sv}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\right)\mathrm{d}v\right),\\
\iint_{\mathbb{R}^2}\ln(t^2+\varepsilon^2)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)&=2\,\mathrm{Im}\left(\int_0^{+\infty}\mathrm{e}^{-\varepsilon v}\left(\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}tv}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\right)\mathrm{d}v\right).\end{aligned}$$

Noting that

$$\iint_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{\mathrm{i}(s-t)v}-1}{\mathrm{i}v}\right)\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)=\frac{1}{\mathrm{i}v}\iint_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}(s-t)v}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)-\frac{1}{\mathrm{i}v}\underbrace{\iint_{\mathbb{R}^2}\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)}_{=0}$$

[10] One could also carry out the proof via the following identity: for $s\in\mathbb{R}$, $n\in\mathbb{N}$, and any $\epsilon>0$, $\ln\left(s^{2+1/n}+\epsilon^2\right)=\ln(\epsilon^2)+2\,\mathrm{Im}\left(\int_0^{+\infty}(\mathrm{i}u)^{-1}(\mathrm{e}^{\mathrm{i}us^{1+1/2n}}-1)\mathrm{e}^{-\epsilon u}\,\mathrm{d}u\right)$.

$$= \frac{1}{\mathrm{i}v} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sv} \, \mathrm{d}\mu(s) \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tv} \, \mathrm{d}\mu(t),$$

and setting $\widehat{\mu}(z) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi z} \, \mathrm{d}\mu(\xi)$, one gets that

$$\iint_{\mathbb{R}^2} \left( \frac{\mathrm{e}^{\mathrm{i}(s-t)v} - 1}{\mathrm{i}v} \right) \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = \frac{1}{\mathrm{i}v} |\widehat{\mu}(v)|^2 :$$

also,

$$\iint_{\mathbb{R}^2} \left( \frac{\mathrm{e}^{\mathrm{i}sv} - 1}{\mathrm{i}v} \right) \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = \frac{1}{\mathrm{i}v} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sv} \, \mathrm{d}\mu(s) \underbrace{\int_{\mathbb{R}} \mathrm{d}\mu(t)}_{=0} - \frac{1}{\mathrm{i}v} \underbrace{\int_{\mathbb{R}} \mathrm{d}\mu(s)}_{=0} \underbrace{\int_{\mathbb{R}} \mathrm{d}\mu(t)}_{=0} = 0;$$

similarly,

$$\iint_{\mathbb{R}^2} \left( \frac{\mathrm{e}^{\mathrm{i}tv} - 1}{\mathrm{i}v} \right) \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = 0.$$

Hence,

$$\left(1 + \frac{1}{2n}\right) \iint_{\mathbb{R}^2} \ln((s-t)^2 + \varepsilon^2) \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = 2\left(1 + \frac{1}{2n}\right) \mathrm{Im}\left( \int_0^{+\infty} \frac{|\widehat{\mu}(v)|^2}{\mathrm{i}v} \mathrm{e}^{-\varepsilon v} \, \mathrm{d}v \right),$$
$$\iint_{\mathbb{R}^2} \ln(s^2 + \varepsilon^2) \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = \iint_{\mathbb{R}^2} \ln(t^2 + \varepsilon^2) \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = 0.$$

Noting that $\widehat{\mu}(0) = \int_{\mathbb{R}} \mathrm{d}\mu(\xi) = 0$, a Taylor expansion about $v = 0$ shows that $\widehat{\mu}(v) =_{v \to 0} \widehat{\mu}'(0)v + O(v^2)$, where $\widehat{\mu}'(0) := \partial_v \widehat{\mu}(v)|_{v=0}$; thus, $v^{-1}|\widehat{\mu}(v)|^2 =_{v \to 0} |\widehat{\mu}'(0)|^2 v + O(v^2)$, which means that there is no singularity in the integrand as $v \to 0$ (in fact, $v^{-1}|\widehat{\mu}(v)|^2$ is real analytic in a neighbourhood of the origin), whence

$$\left(1 + \frac{1}{2n}\right) \iint_{\mathbb{R}^2} \ln((s-t)^2 + \varepsilon^2) \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = -2\left(1 + \frac{1}{2n}\right) \int_0^{+\infty} v^{-1} |\widehat{\mu}(v)|^2 \mathrm{e}^{-\varepsilon v} \, \mathrm{d}v.$$

Recalling that $\iint_{\mathbb{R}^2} \ln(*^2 + \varepsilon^2) \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = 0$, $* \in \{s, t\}$, and adding, it follows that

$$\iint_{\mathbb{R}^2} \ln\left( \frac{(s^2+\varepsilon^2)^{1/2} (t^2+\varepsilon^2)^{1/2}}{((s-t)^2+\varepsilon^2)^{1+\frac{1}{2n}}} \right) \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = 2\left(1 + \frac{1}{2n}\right) \int_0^{+\infty} v^{-1} |\widehat{\mu}(v)|^2 \mathrm{e}^{-\varepsilon v} \, \mathrm{d}v.$$

Now, using the fact that $\ln((s-t)^2+\varepsilon^2)^{-1}$ (resp., $\ln(s^2+\varepsilon^2)^{1/2}$ and $\ln(t^2+\varepsilon^2)^{1/2}$) is (resp., are) bounded below (resp., above) uniformly with respect to $\varepsilon$ and that the measures have compact support, letting $\varepsilon \downarrow 0$ and using the Monotone Convergence Theorem, one arrives at

$$\begin{aligned} \left(1 + \frac{1}{2n}\right) \iint_{\mathbb{R}^2} \ln\left( \frac{(s^2+\varepsilon^2)^{1/2} (t^2+\varepsilon^2)^{1/2}}{((s-t)^2+\varepsilon^2)^{1+\frac{1}{2n}}} \right) \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) &\underset{\varepsilon \downarrow 0}{=} \iint_{\mathbb{R}^2} \ln\left( \frac{|st|}{|s-t|^{2+\frac{1}{n}}} \right) \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) \\ &= 2\left(1 + \frac{1}{2n}\right) \int_0^{+\infty} v^{-1} |\widehat{\mu}(v)|^2 \, \mathrm{d}v \geqslant 0, \end{aligned}$$

where, trivially, equality holds if, and only if, $\mu = 0$. Furthermore, noting that, since $\int_{\mathbb{R}} \mathrm{d}\mu(\xi) = 0$, $\iint_{\mathbb{R}^2} \ln(w^e(*))^{-1} \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = 0$, $* \in \{s, t\}$, letting $\varepsilon \downarrow 0$ and using monotone convergence, one also arrives at

$$\begin{aligned} \iint_{\mathbb{R}^2} \ln\left( \frac{(s^2+\varepsilon^2)^{1/2} (t^2+\varepsilon^2)^{1/2}}{((s-t)^2+\varepsilon^2)^{1+\frac{1}{2n}} w^o(s) w^o(t)} \right) \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) &\underset{\varepsilon \downarrow 0}{=} \iint_{\mathbb{R}^2} \ln\left( \frac{|st|}{|s-t|^{2+\frac{1}{n}} w^o(s) w^o(t)} \right) \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) \\ &= 2\left(1 + \frac{1}{2n}\right) \int_0^{+\infty} v^{-1} |\widehat{\mu}(v)|^2 \, \mathrm{d}v \geqslant 0, \end{aligned}$$

where, again, and trivially, equality holds if, and only if, $\mu = 0$. □

The uniqueness of $\mu_V^o$ $(\in \mathcal{M}_1(\mathbb{R}))$ will now be established.

**Lemma 3.3.** *Let the external field* $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5)*. Set* $w^o(z):=\exp(-\widetilde{V}(z))$*, and define, for* $n\in\mathbb{N}$*,*

$$\mathrm{I}_V^o[\mu^o]\colon\ \mathcal{M}_1(\mathbb{R})\to\mathbb{R},\ \ \mu^o\mapsto\iint_{\mathbb{R}^2}\ln\Big(|s-t|^{2+\frac{1}{n}}|st|^{-1}w^o(s)w^o(t)\Big)^{-1}\,\mathrm{d}\mu^o(s)\,\mathrm{d}\mu^o(t),$$

*and consider the minimisation problem* $E_V^o=\inf\Big\{\mathrm{I}_V^o[\mu^o];\ \mu^o\in\mathcal{M}_1(\mathbb{R})\Big\}$*. Then,* $\exists!\ \mu_V^o\in\mathcal{M}_1(\mathbb{R})$ *such that* $\mathrm{I}_V^o[\mu_V^o]=E_V^o$*.*

*Proof.* It was shown in Lemma 3.1 that $\exists\ \mu_V^o\in\mathcal{M}_1(\mathbb{R})$, the 'odd' equilibrium measure, such that $\mathrm{I}_V^o[\mu^o]=E_V^o$; therefore, it remains to establish the uniqueness of the 'odd' equilibrium measure. Let $\widetilde{\mu}_V^o\in\mathcal{M}_1(\mathbb{R})$ be a second probability measure for which $\mathrm{I}_V^o[\widetilde{\mu}_V^o]=E_V^o=\mathrm{I}_V^o[\mu_V^o]$: the argument in Lemma 3.1 shows that $\widetilde{J}_o:=\mathrm{supp}(\widetilde{\mu}_V^o)=\text{compact}\subset\overline{\mathbb{R}}\setminus\{0,\pm\infty\}$, and that $\mathrm{I}_V^o[\widetilde{\mu}_V^o]<+\infty$. Define the finite-moment, signed measure $\mu^\sharp:=\widetilde{\mu}_V^o-\mu_V^o$, where $\widetilde{\mu}_V^o,\mu_V^o\in\mathcal{M}_1(\mathbb{R})$, and $\widetilde{J}_o\cap J_o=\varnothing$, with (cf. Lemma 3.1), $J_o=\mathrm{supp}(\mu_V^o)=\text{compact}\subset\overline{\mathbb{R}}\setminus\{0,\pm\infty\}$; thus, from Lemma 3.2 (with $\mu\to\mu^\sharp$), namely,

$$\iint_{\mathbb{R}^2}\ln\Big(|s-t|^{-(2+\frac{1}{n})}|st|\Big)\,\mathrm{d}\mu^\sharp(s)\,\mathrm{d}\mu^\sharp(t)=\iint_{\mathbb{R}^2}\ln\Big(|s-t|^{2+\frac{1}{n}}|st|^{-1}w^o(s)w^o(t)\Big)^{-1}\,\mathrm{d}\mu^\sharp(s)\,\mathrm{d}\mu^\sharp(t)\geqslant 0,$$

it follows that

$$\begin{aligned}\iint_{\mathbb{R}^2}\ln\bigg(\frac{|st|}{|s-t|^{2+\frac{1}{n}}}\bigg)\Big(\mathrm{d}\widetilde{\mu}_V^o(s)\,\mathrm{d}\widetilde{\mu}_V^o(t)+\mathrm{d}\mu_V^o(s)\,\mathrm{d}\mu_V^o(t)\Big)\geqslant\ &\iint_{\mathbb{R}^2}\ln\bigg(\frac{|st|}{|s-t|^{2+\frac{1}{n}}}\bigg)\Big(\mathrm{d}\widetilde{\mu}_V^o(s)\,\mathrm{d}\mu_V^o(t)\\&+\ \mathrm{d}\mu_V^o(s)\mathrm{d}\widetilde{\mu}_V^o(t)\Big),\end{aligned}$$

or, via a straightforward symmetry argument,

$$\begin{aligned}\iint_{\mathbb{R}^2}\ln\bigg(\frac{|st|}{|s-t|^{2+\frac{1}{n}}}\bigg)\Big(\mathrm{d}\widetilde{\mu}_V^o(s)\,\mathrm{d}\widetilde{\mu}_V^o(t)+\mathrm{d}\mu_V^o(s)\,\mathrm{d}\mu_V^o(t)\Big)&\geqslant 2\iint_{\mathbb{R}^2}\ln\bigg(\frac{|st|}{|s-t|^{2+\frac{1}{n}}}\bigg)\mathrm{d}\widetilde{\mu}_V^o(s)\,\mathrm{d}\mu_V^o(t)\\&=2\iint_{\mathbb{R}^2}\ln\bigg(\frac{|st|}{|s-t|^{2+\frac{1}{n}}}\bigg)\mathrm{d}\mu_V^o(s)\,\mathrm{d}\widetilde{\mu}_V^o(t).\end{aligned}$$

The above shows that (since both $\mathrm{I}_V^o[\mu_V^o]$ and $\mathrm{I}_V^o[\widetilde{\mu}_V^o]<+\infty$) $\ln(|st||s-t|^{-(2+\frac{1}{n})})$ is integrable with respect to both $\mathrm{d}\widetilde{\mu}_V^o(s)\,\mathrm{d}\mu_V^o(t)$ and $\mathrm{d}\mu_V^o(s)\,\mathrm{d}\widetilde{\mu}_V^o(t)$. From an argument on pg. 149 of [79], it follows that $\ln(|st||s-t|^{-(2+\frac{1}{n})})$ is integrable with respect to (the measure) $\mathrm{d}\mu_t^o(s)\,\mathrm{d}\mu_t^o(t')$, where $\mu_t^o(z):=\mu_V^o(z)+t(\widetilde{\mu}_V^o(z)-\mu_V^o(z))$, $(z,t)\in\mathbb{R}\times[0,1]$. Set

$$\mathcal{F}_\mu(t):=\iint_{\mathbb{R}^2}\ln\Big(|st'||s-t'|^{-(2+\frac{1}{n})}(w^o(s)w^o(t'))^{-1}\Big)\,\mathrm{d}\mu_t^o(s)\,\mathrm{d}\mu_t^o(t')$$

($=\mathrm{I}_V^o[\mu_t^o]$). Noting that

$$\begin{aligned}\mathrm{d}\mu_t^o(s)\,\mathrm{d}\mu_t^o(t')&=\mathrm{d}\mu_V^o(s)\,\mathrm{d}\mu_V^o(t')+t\mathrm{d}\mu_V^o(s)(\mathrm{d}\widetilde{\mu}_V^o(t')-\mathrm{d}\mu_V^o(t'))+t\mathrm{d}\mu_V^o(t')(\mathrm{d}\widetilde{\mu}_V^o(s)-\mathrm{d}\mu_V^o(s))\\&\quad+t^2(\mathrm{d}\widetilde{\mu}_V^o(s)-\mathrm{d}\mu_V^o(s))(\mathrm{d}\widetilde{\mu}_V^o(t')-\mathrm{d}\mu_V^o(t')),\end{aligned}$$

it follows that

$$\begin{aligned}\mathcal{F}_\mu(t)&=\mathrm{I}_V^o[\mu_V^o]+2t\iint_{\mathbb{R}^2}\ln\bigg(\frac{|st'|}{|s-t'|^{2+\frac{1}{n}}}(w^o(s)w^o(t'))^{-1}\bigg)\mathrm{d}\mu_V^o(s)\,(\mathrm{d}\widetilde{\mu}_V^o(t')-\mathrm{d}\mu_V^o(t'))\\&\quad+t^2\iint_{\mathbb{R}^2}\ln\bigg(\frac{|st'|}{|s-t'|^{2+\frac{1}{n}}}(w^o(s)w^o(t'))^{-1}\bigg)(\mathrm{d}\widetilde{\mu}_V^o(s)-\mathrm{d}\mu_V^o(s))\,(\mathrm{d}\widetilde{\mu}_V^o(t')-\mathrm{d}\mu_V^o(t')).\end{aligned}$$

Since $\mu^\sharp\in\mathcal{M}_1(\mathbb{R})$ is finite-moment signed measure with mean zero, that is, $\int_{\mathbb{R}}\mathrm{d}\mu^\sharp(\xi)=\int_{\mathbb{R}}\mathrm{d}(\widetilde{\mu}_V^o-\mu_V^o)(\xi)=0$, and compact support, it follows from the analysis above and the result of Lemma 3.2 that $\mathcal{F}_\mu(t)$ is convex[11]; thus, for $t\in[0,1]$,

$$\mathrm{I}_V^o[\mu_V^o]\leqslant\mathcal{F}_\mu(t)=\mathrm{I}_V^o[\mu_t^o]=\mathcal{F}_\mu(t+(1-t)0)\leqslant t\mathcal{F}_\mu(1)+(1-t)\mathcal{F}_\mu(0)$$

[11]If $f$ is twice differentiable on $(a,b)$, then $f''(x)\geqslant 0$ on $(a,b)$ is both a necessary and sufficient condition that $f$ be convex on $(a,b)$.

$$= t\mathrm{I}_V^o[\widetilde{\mu}_V^o]+(1-t)\mathrm{I}_V^o[\mu_V^o]=t\mathrm{I}_V^o[\mu_V^o]+(1-t)\mathrm{I}_V^o[\mu_V^o] \Rightarrow$$
$$\mathrm{I}_V^o[\mu_V^o] \leqslant \mathrm{I}_V^o[\mu_t^o] \leqslant \mathrm{I}_V^o[\mu_V^o],$$

whence $\mathrm{I}_V^o[\mu_t^o] = \mathrm{I}_V^o[\mu_V^o] := E_V^o$ (= const.). Since $\mathrm{I}_V^o[\mu_t^o] = \mathcal{F}_\mu(t) = E_V^o$, it follows, in particular, that $\mathcal{F}_\mu''(0)=0 \Rightarrow$

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^2} \ln\left(\frac{|st'|}{|s-t'|^{2+\frac{1}{n}}}(w^o(s)w^o(t'))^{-1}\right)(\mathrm{d}\widetilde{\mu}_V^o(s)-\mathrm{d}\mu_V^o(s))\,(\mathrm{d}\widetilde{\mu}_V^o(t')-\mathrm{d}\mu_V^o(t')) \\
&= \iint_{\mathbb{R}^2} \ln\left(\frac{|st'|}{|s-t'|^{2+\frac{1}{n}}}\right)(\mathrm{d}\widetilde{\mu}_V^o(s)-\mathrm{d}\mu_V^o(s))\,(\mathrm{d}\widetilde{\mu}_V^o(t')-\mathrm{d}\mu_V^o(t')) \\
&\quad + 2\int_{\mathbb{R}} \widetilde{V}(t')\,\mathrm{d}(\widetilde{\mu}_V^o-\mu_V^o)(t')\,\underbrace{\int_{\mathbb{R}} \mathrm{d}(\widetilde{\mu}_V^o-\mu_V^o)(s)}_{=0} \Rightarrow \\
0 &= \iint_{\mathbb{R}^2} \ln\left(\frac{|st'|}{|s-t'|^{2+\frac{1}{n}}}\right)\mathrm{d}(\widetilde{\mu}_V^o-\mu_V^o)(s)\,\mathrm{d}(\widetilde{\mu}_V^o-\mu_V^o)(t');
\end{aligned}$$

but, in Lemma 3.2, it was shown that,

$$\iint_{\mathbb{R}^2} \ln\left(\frac{|st'|}{|s-t'|^{2+\frac{1}{n}}}\right)\mathrm{d}(\widetilde{\mu}_V^o-\mu_V^o)(s)\,\mathrm{d}(\widetilde{\mu}_V^o-\mu_V^o)(t')=\left(2+\frac{1}{n}\right)\int_0^{+\infty} \xi^{-1}|(\widehat{\widetilde{\mu}_V^o}-\widehat{\mu_V^o})(\xi)|^2\,\mathrm{d}\xi \quad (\geqslant 0),$$

whence $\int_0^{+\infty} \xi^{-1}|(\widehat{\widetilde{\mu}_V^o}-\widehat{\mu_V^o})(\xi)|^2\,\mathrm{d}\xi=0 \Rightarrow \widehat{\widetilde{\mu}_V^o}(\xi)=\widehat{\mu_V^o}(\xi)$, $\xi \geqslant 0$. Noting that

$$\widehat{\widetilde{\mu}_V^o}(-\xi)=\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}s(-\xi)}\,\mathrm{d}\widetilde{\mu}_V^o(s)=\overline{\widehat{\widetilde{\mu}_V^o}(\xi)} \qquad \text{and} \qquad \widehat{\mu_V^o}(-\xi)=\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}s(-\xi)}\,\mathrm{d}\mu_V^o(s)=\overline{\widehat{\mu_V^o}(\xi)},$$

it follows from $\widehat{\widetilde{\mu}_V^o}(\xi)=\widehat{\mu_V^o}(\xi)$, $\xi \geqslant 0$, via a complex-conjugation argument, that $\widehat{\widetilde{\mu}_V^o}(-\xi)=\widehat{\mu_V^o}(-\xi)$, $\xi \geqslant 0$; hence, $\widehat{\widetilde{\mu}_V^o}(\xi)=\widehat{\mu_V^o}(\xi)$, $\xi \in \mathbb{R}$. The latter relation shows that $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}(\widetilde{\mu}_V^o-\mu_V^o)(s)=0 \Rightarrow \widetilde{\mu}_V^o=\mu_V^o$; thus the uniqueness of the 'odd' equilibrium measure. □

Before proceeding to Lemma 3.4, the following observations, which are interesting, non-trivial and important results in their own right, should be noted. Let $\widetilde{V} \colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ satisfy conditions (2.3)–(2.5). For each $m \in \mathbb{Z}_0^+$ and any $(2m+1)$-tuple $(x_1, x_2, \dots, x_{2m+1})$ of distinct, finite and non-zero real numbers, let, for $n \in \mathbb{N}$,

$$\mathfrak{d}_{o,m}^{\widetilde{V}}(n) := \frac{1}{2m(2m+1)} \inf_{\{x_1,x_2,\dots,x_{2m+1}\}\subset\mathbb{R}\setminus\{0\}} \left(\sum_{\substack{j,k=1\\ j\neq k}}^{2m+1} \ln\left(\left|x_j-x_k\right|^{1+\frac{1}{n}}\left|x_k^{-1}-x_j^{-1}\right|\right)^{-1} + 4m\sum_{i=1}^{2m+1} \widetilde{V}(x_i)\right).$$

For each $m \in \mathbb{Z}_0^+$, a set $\left\{x_1^\flat, x_2^\flat, \dots, x_{2m+1}^\flat\right\}$ which realizes the above infimum, that is, for $n \in \mathbb{N}$,

$$\mathfrak{d}_{o,m}^{\widetilde{V}}(n) = \frac{1}{2m(2m+1)}\left(\sum_{\substack{j,k=1\\ j\neq k}}^{2m+1} \ln\left(\left|x_j^\flat-x_k^\flat\right|^{1+\frac{1}{n}}\left|(x_k^\flat)^{-1}-(x_j^\flat)^{-1}\right|\right)^{-1} + 4m\sum_{i=1}^{2m+1} \widetilde{V}(x_i^\flat)\right),$$

will be called (with slight abuse of nomenclature) a *generalised weighted* $(2m+1)$-*Fekete set*, and the points $x_1^\flat, x_2^\flat, \dots, x_{2m+1}^\flat$ will be called *generalised weighted Fekete points*. For $\left\{x_1^\flat, x_2^\flat, \dots, x_{2m+1}^\flat\right\}$ a generalised weighted $(2m+1)$-Fekete set, denote by

$$\mu_{\mathbf{x}^\flat}^o := \frac{1}{2m+1}\sum_{j=1}^{2m+1} \delta_{x_j^\flat},$$

where $\delta_{x_j^\flat}$, $j=1,\dots,2m+1$, is the Dirac delta measure (atomic mass) concentrated at $x_j^\flat$, the *normalised counting measure*, that is, $\int_{\mathbb{R}} \mathrm{d}\mu_{\mathbf{x}^\flat}^o(s)=1$. Then, mimicking the calculations in Chapter 6 of [79] and the techniques used to prove Theorem 1.34 in [44] (see, in particular, Section 2 of [44]), one proves that, for $n \in \mathbb{N}$ (the details are left to the interested reader):

- $\lim_{m\to\infty}\mathfrak{d}_{o,m}^{\widetilde{V}}(n)$ exists, more precisely,

$$\lim_{m\to\infty}\mathfrak{d}_{o,m}^{\widetilde{V}}(n)=E_V^o=\inf\left\{\mathrm{I}_V^o[\mu^o];\ \mu^o\in\mathcal{M}_1(\mathbb{R})\right\},$$

where (the functional) $\mathrm{I}_V^o[\mu^o]\colon\ \mathcal{M}_1(\mathbb{R})\to\mathbb{R}$ is defined in Lemma 3.1, and $\lim_{m\to\infty}\exp(-\mathfrak{d}_{o,m}^{\widetilde{V}}(n))=\exp(-E_V^o)$ is positive and finite;
- $\mu_{\mathbf{x}^\flat}^o$ converges weakly (in the weak-$*$ topology of measures) to the 'odd' equilibrium measure $\mu_V^o$, that is, $\mu_{\mathbf{x}^\flat}^o\overset{*}{\to}\mu_V^o$ as $m\to\infty$.

**RHP2**, that is, $(\overset{o}{\mathrm{Y}}(z),\mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_+,\mathbb{R})$, is now reformulated as an equivalent, auxiliary RHP normalised at zero.

**Notational Remark 3.1.** For completeness, the integrand appearing in the definition of $g^o(z)$ (see Lemma 3.4 below) is defined as follows: $\ln((z-s)^{2+\frac{1}{n}}(zs)^{-1}):=(2+\frac{1}{n})\ln(z-s)-\ln z-\ln s$, where, for $s<0$, $\ln s:=\ln|s|+\mathrm{i}\pi$. ■

**Lemma 3.4.** *Let the external field* $\widetilde{V}\colon\ \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5). *For the associated 'odd' equilibrium measure,* $\mu_V^o\in\mathcal{M}_1(\mathbb{R})$, *set* $J_o:=\operatorname{supp}(\mu_V^o)$, *where* $J_o$ *(= compact)* $\subset\overline{\mathbb{R}}\setminus\{0,\pm\infty\}$, *and let* $\overset{o}{\mathrm{Y}}\colon\ \mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ *be the (unique) solution of* **RHP2**. *Let*

$$\overset{o}{\mathcal{M}}(z):=\mathrm{e}^{-\frac{n\ell_o}{2}\operatorname{ad}(\sigma_3)}\overset{o}{\mathrm{Y}}(z)\mathrm{e}^{-n(g^o(z)-\mathfrak{Q}_{\mathcal{A}})\sigma_3},$$

*where* $g^o(z)$, *the 'odd' g-function, is defined by, for* $n\in\mathbb{N}$,

$$g^o(z):=\int_{J_o}\ln\Big((z-s)^{2+\frac{1}{n}}(zs)^{-1}\Big)\,\mathrm{d}\mu_V^o(s),\quad z\in\mathbb{C}\setminus(-\infty,\max\{0,\max\{\operatorname{supp}(\mu_V^o)\}\}),$$

$\ell_o$ $(\in\mathbb{R})$, *the 'odd' variational constant, is given in* Lemma 3.6 *below, and*

$$\mathfrak{Q}_{\mathcal{A}}=\begin{cases}\mathfrak{Q}_{\mathcal{A}}^+:=(1+\frac{1}{n})\int_{J_o}\ln(|s|)\,\mathrm{d}\mu_V^o(s)-\mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\mathrm{d}\mu_V^o(s)+\mathrm{i}\pi(2+\frac{1}{n})\int_{J_o\cap\mathbb{R}_+}\mathrm{d}\mu_V^o(s), & z\in\mathbb{C}_+,\\ \mathfrak{Q}_{\mathcal{A}}^-:=(1+\frac{1}{n})\int_{J_o}\ln(|s|)\,\mathrm{d}\mu_V^o(s)-\mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\mathrm{d}\mu_V^o(s)-\mathrm{i}\pi(2+\frac{1}{n})\int_{J_o\cap\mathbb{R}_+}\mathrm{d}\mu_V^o(s), & z\in\mathbb{C}_-,\end{cases}$$

*with (see Lemma* 3.5, *item* (1), *below)*

$$\int_{J_o\cap\mathbb{R}_-}\mathrm{d}\mu_V^o(s)=\begin{cases}0, & J_o\subset\mathbb{R}_+,\\ 1, & J_o\subset\mathbb{R}_-,\\ \int_{b_0^o}^{a_j^o}\mathrm{d}\mu_V^o(s), & (a_j^o,b_j^o)\ni 0,\quad j=1,\dots,N,\end{cases}$$

*and*

$$\int_{J_o\cap\mathbb{R}_+}\mathrm{d}\mu_V^o(s)=\begin{cases}0, & J_o\subset\mathbb{R}_-,\\ 1, & J_o\subset\mathbb{R}_+,\\ \int_{b_j^o}^{a_{N+1}^o}\mathrm{d}\mu_V^o(s), & (a_j^o,b_j^o)\ni 0,\quad j=1,\dots,N.\end{cases}$$

*Then* $\overset{o}{\mathcal{M}}\colon\ \mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ *solves the following (normalised at zero)* RHP: (i) $\overset{o}{\mathcal{M}}(z)$ *is holomorphic for* $z\in\mathbb{C}\setminus\mathbb{R}$; (ii) *the boundary values* $\overset{o}{\mathcal{M}}_\pm(z):=\lim_{\substack{z'\to z\\ \pm\mathrm{Im}(z')>0}}\overset{o}{\mathcal{M}}(z')$ *satisfy the jump condition*

$$\overset{o}{\mathcal{M}}_+(z)=\overset{o}{\mathcal{M}}_-(z)\begin{pmatrix}\mathrm{e}^{-n(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^++\mathfrak{Q}_{\mathcal{A}}^-)} & \mathrm{e}^{n(g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+-\mathfrak{Q}_{\mathcal{A}}^-)}\\ 0 & \mathrm{e}^{n(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^++\mathfrak{Q}_{\mathcal{A}}^-)}\end{pmatrix},\quad z\in\mathbb{R},$$

*with* $g_\pm^o(z):=\lim_{\varepsilon\downarrow 0}g^o(z\pm\mathrm{i}\varepsilon)$; (iii) $\overset{o}{\mathcal{M}}(z)\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathrm{I}+\mathcal{O}(z)$; *and* (iv) $\overset{o}{\mathcal{M}}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathcal{O}(1)$.

*Proof.* For (arbitrary) $z_1,z_2\in\mathbb{C}_\pm$, note that, from the definition of $g^o(z)$ stated in the Lemma, $g^o(z_2)-g^o(z_1)=\mathrm{i}\pi\int_{z_1}^{z_2}\mathcal{F}^o(s)\,\mathrm{d}s$, where

$$\mathcal{F}^o\colon\ \mathbb{C}\setminus(\operatorname{supp}(\mu_V^o)\cup\{0\})\to\mathbb{C},\ z\mapsto-\frac{1}{\mathrm{i}\pi}\left(\frac{1}{z}+\left(2+\frac{1}{n}\right)\int_{J_o}\frac{\mathrm{d}\mu_V^o(s)}{s-z}\right),$$

with $\mathcal{F}^{o}(z) =_{z\to 0} -\frac{1}{\pi \mathrm{i} z} + \mathcal{O}(1)$ (since $\mu_V^o \in \mathcal{M}_1(\mathbb{R})$; in particular, $\int_{\mathbb{R}} s^m \,\mathrm{d}\mu_V^o(s) < \infty$, $m \in \mathbb{Z}$); thus, $|g^o(z_2) - g^o(z_1)| \leqslant \pi \sup_{z \in \mathbb{C}_\pm} |\mathcal{F}^o(z)||z_2 - z_1|$, that is, $g^o(z)$ is uniformly Lipschitz continuous in $\mathbb{C}_\pm$. Thus, from the definition of $g^o(z)$ stated in the Lemma:

(1) for $s \in J_o$, $z \in \mathbb{C} \setminus (-\infty, \max\{0, \max\{\operatorname{supp}(\mu_V^o)\}\})$, with $|s/z| \ll 1$, and $\mu_V^o \in \mathcal{M}_1(\mathbb{R})$, in particular, $\int_{\mathbb{R}} \mathrm{d}\mu_V^o(s)$ $(= \int_{J_o} \mathrm{d}\mu_V^o(s)) = 1$ and $\int_{\mathbb{R}} s^m \,\mathrm{d}\mu_V^o(s)$ $(= \int_{J_o} s^m \,\mathrm{d}\mu_V^o(s)) < \infty$, $m \in \mathbb{N}$, it follows from the expansions $\frac{1}{s-z} = -\sum_{k=0}^{l} \frac{s^k}{z^{k+1}} + \frac{s^{l+1}}{z^{l+1}(s-z)}$, $l \in \mathbb{Z}_0^+$, and $\ln(z-s) =_{|z|\to\infty} \ln(z) - \sum_{k=1}^{\infty} \frac{1}{k}(\frac{s}{z})^k$, that

$$g^o(z) \underset{\mathbb{C}\setminus\mathbb{R}\ni z\to\infty}{=} \left(1+\frac{1}{n}\right)\ln(z) - \int_{J_o} \ln(|s|)\,\mathrm{d}\mu_V^o(s) - \mathrm{i}\pi \int_{J_o\cap\mathbb{R}_-} \mathrm{d}\mu_V^o(s) + \mathcal{O}(z^{-1}),$$

where $\int_{J_o\cap\mathbb{R}_-} \mathrm{d}\mu_V^o(s)$ is given in the Lemma;

(2) for $s \in J_o$, $z \in \mathbb{C} \setminus (-\infty, \max\{0, \max\{\operatorname{supp}(\mu_V^o)\}\})$, with $|z/s| \ll 1$, and $\mu_V^o \in \mathcal{M}_1(\mathbb{R})$, in particular, $\int_{\mathbb{R}} s^{-m} \,\mathrm{d}\mu_V^o(s)$ $(= \int_{J_o} s^{-m} \,\mathrm{d}\mu_V^o(s)) < \infty$, $m \in \mathbb{N}$, it follows from the expansions $\frac{1}{z-s} = -\sum_{k=0}^{l} \frac{z^k}{s^{k+1}} + \frac{z^{l+1}}{s^{l+1}(z-s)}$, $l \in \mathbb{Z}_0^+$, and $\ln(s-z) =_{|z|\to 0} \ln(s) - \sum_{k=1}^{\infty} \frac{1}{k}(\frac{z}{s})^k$, that

$$\begin{aligned} g^o(z) \underset{\mathbb{C}_\pm\ni z\to 0}{=} &-\ln(z) + \left(1+\frac{1}{n}\right)\int_{J_o} \ln(|s|)\,\mathrm{d}\mu_V^o(s) - \mathrm{i}\pi \int_{J_o\cap\mathbb{R}_-} \mathrm{d}\mu_V^o(s) \\ &\pm \mathrm{i}\pi\left(2+\frac{1}{n}\right)\int_{J_o\cap\mathbb{R}_+} \mathrm{d}\mu_V^o(s) + \mathcal{O}(z), \end{aligned}$$

where $\int_{J_o\cap\mathbb{R}_+} \mathrm{d}\mu_V^o(s)$ is given in the Lemma.

Items (i)–(iv) now follow from the definitions of $\overset{o}{\mathcal{M}}(z)$ (in terms of $\overset{o}{\mathrm{Y}}(z)$) and $g^o(z)$ stated in the Lemma, and the above two asymptotic expansions. □

**Lemma 3.5.** *Let the external field* $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfy conditions* (2.3)–(2.5). *For* $\mu_V^o \in \mathcal{M}_1(\mathbb{R})$, *the associated 'odd' equilibrium measure, set* $J_o := \operatorname{supp}(\mu_V^o)$, *where* $J_o$ (= *compact*) $\subset \overline{\mathbb{R}} \setminus \{0, \pm\infty\}$. *Then:* (1) $J_o = \cup_{j=1}^{N+1}(b_{j-1}^o, a_j^o)$, *with* $N \in \mathbb{N}$ *and finite,* $b_0^o := \min\{\operatorname{supp}(\mu_V^o)\} \notin \{-\infty, 0\}$, $a_{N+1}^o := \max\{\operatorname{supp}(\mu_V^o)\} \notin \{0, +\infty\}$, *and* $-\infty < b_0^o < a_1^o < b_1^o < a_2^o < \cdots < b_N^o < a_{N+1}^o < +\infty$, *and* $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$ *satisfy the n-dependent and (locally) solvable system of* $2(N+1)$ *moment conditions*

$$\int_{J_o} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^j}{(R_o(s))_+^{1/2}} \frac{\mathrm{d}s}{2\pi\mathrm{i}} = 0, \quad j = 0, \dots, N, \qquad \int_{J_o} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+1}}{(R_o(s))_+^{1/2}} \frac{\mathrm{d}s}{2\pi\mathrm{i}} = \frac{1}{\mathrm{i}\pi}\left(2+\frac{1}{n}\right),$$

$$\int_{a_j^o}^{b_j^o} \left(\mathrm{i}(R_o(s))^{1/2} \int_{J_o} \frac{(\frac{\mathrm{i}}{\pi\xi} + \frac{\mathrm{i}\widetilde{V}'(\xi)}{2\pi})}{(R_o(\xi))_+^{1/2}(\xi - s)} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\right)\mathrm{d}s = \frac{1}{2\pi} \ln\left|\frac{a_j^o}{b_j^o}\right| + \frac{1}{4\pi}\left(\widetilde{V}(a_j^o) - \widetilde{V}(b_j^o)\right), \quad j = 1, \dots, N,$$

*where* $(R_o(z))^{1/2}$ *is defined in Theorem* 2.3.1, *Equation* (2.8), *with* $(R_o(z))_\pm^{1/2} := \lim_{\varepsilon\downarrow 0}(R_o(z \pm \mathrm{i}\varepsilon))^{1/2}$, *and the branch of the square root chosen so that* $z^{-(N+1)}(R_o(z))^{1/2} \sim_{\substack{z\to\infty \\ z\in\mathbb{C}_\pm}} \pm 1$; *and* (2) *the density of the 'odd' equilibrium measure, which is absolutely continuous with respect to Lebesgue measure, is given by*

$$\mathrm{d}\mu_V^o(x) := \psi_V^o(x)\,\mathrm{d}x = \frac{1}{2\pi\mathrm{i}}(R_o(x))_+^{1/2} h_V^o(x) \mathbf{1}_{J_o}(x)\,\mathrm{d}x,$$

*where*

$$h_V^o(z) := \frac{1}{2}\left(2+\frac{1}{n}\right)^{-1} \oint_{C_{\mathrm{R}}^o} \frac{(\frac{2\mathrm{i}}{\pi s} + \frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_o(s))^{1/2}(s-z)}\,\mathrm{d}s$$

*(real analytic for* $z \in \mathbb{R} \setminus \{0\}$*), with* $C_{\mathrm{R}}^o$ $(\subset \mathbb{C}^*)$ *the boundary of any open doubly-connected annular region of the type* $\{z' \in \mathbb{C};\ 0 < r < |z'| < R < +\infty\}$, *where the simple outer (resp., inner) boundary* $\{z' = R\mathrm{e}^{\mathrm{i}\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi\}$ *(resp.,* $\{z' = r\mathrm{e}^{\mathrm{i}\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi\}$*) is traversed clockwise (resp., counter-clockwise), with the numbers* $0 < r < R < +\infty$ *chosen such that, for (any) non-real z in the domain of analyticity of* $\widetilde{V}$ *(that is,* $\mathbb{C}^*$*),* $\operatorname{int}(C_{\mathrm{R}}^o) \supset J_o \cup \{z\}$, $\mathbf{1}_{J_o}(x)$ *is the indicator (characteristic) function of the set* $J_o$, *and* $\psi_V^o(x) \geqslant 0$ *(resp.,* $\psi_V^o(x) > 0$*)* $\forall\ x \in \overline{J_o} := \cup_{j=1}^{N+1}[b_{j-1}^o, a_j^o]$ *(resp.,* $\forall\ x \in J_o$*).*

*Proof.* One begins by showing that the support of the 'odd' equilibrium measure, $\mathrm{supp}(\mu_V^o) =: J_o$, consists of the union of a finite number of disjoint and bounded (real) intervals. Recall from Lemma 3.1 that $J_o =$ compact $\subset \overline{\mathbb{R}} \setminus \{0, \pm\infty\}$, and that $\widetilde{V}$ is real analytic on $\mathbb{R} \setminus \{0\}$, thus real analytic on $J_o$, with an analytic continuation to the following (open) neighbourhood of $J_o$, $\mathbb{U} := \{z \in \mathbb{C};\ \inf_{q \in J_o} |z-q| < r \in (0,1)\} \setminus \{0\}$. In analogy with Equation (2.1) of [44], for each $m \in \mathbb{Z}_0^+$ and any $2m+1$-tuple $(x_1, x_2, \dots, x_{2m+1})$ of distinct, finite and non-zero real numbers, let, for $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathrm{d}_{\widetilde{V},m}^o(n) &:= \left( \sup_{\{x_1,x_2,\dots,x_{2m+1}\} \subset \mathbb{R}\setminus\{0\}} \prod_{\substack{j,k=1 \\ j<k}}^{2m+1} \left| x_j - x_k \right|^{2+\frac{2}{n}} \left| x_k^{-1} - x_j^{-1} \right|^2 \mathrm{e}^{-2\widetilde{V}(x_j)} \mathrm{e}^{-2\widetilde{V}(x_k)} \right)^{\frac{1}{2m(2m+1)}} \\
&= \left( \sup_{\{x_1,x_2,\dots,x_{2m+1}\} \subset \mathbb{R}\setminus\{0\}} \prod_{\substack{j,k=1 \\ j<k}}^{2m+1} \left| x_j - x_k \right|^{2+\frac{2}{n}} \left| x_k^{-1} - x_j^{-1} \right|^2 \mathrm{e}^{-4m \sum_{i=1}^{2m+1} \widetilde{V}(x_i)} \right)^{\frac{1}{2m(2m+1)}},
\end{aligned}
$$

where $\prod_{\substack{j,k=1 \\ j<k}}^{2m+1}(\star) = \prod_{j=1}^{2m} \prod_{k=j+1}^{2m+1}(\star)$. Denote by $\left\{x_1^*, x_2^*, \dots, x_{2m+1}^*\right\}$, with $x_i^* < x_j^*\ \forall\ i<j \in \{1, \dots, 2m+1\}$, the associated generalised weighted $(2m+1)$-Fekete set (see the discussion preceding Lemma 3.4), that is, for $n \in \mathbb{N}$,

$$
\mathrm{d}_{\widetilde{V},m}^o(n) = \left( \prod_{\substack{j,k=1 \\ j<k}}^{2m+1} \left| x_j^* - x_k^* \right|^{2+\frac{2}{n}} \left| (x_k^*)^{-1} - (x_j^*)^{-1} \right|^2 \mathrm{e}^{-4m \sum_{i=1}^{2m+1} \widetilde{V}(x_i^*)} \right)^{\frac{1}{2m(2m+1)}}.
$$

Proceeding, now, as in the proof of Theorem 1.34, Equation (1.35), of [44], in particular, mimicking the calculations on pp. 408–413 of [44] (for the proofs of Lemmae 2.3 and 2.15 therein), namely, using those techniques to show that, in the present case, the nearest-neighbour distances $\{x_{j+1}^* - x_j^*\}_{j=1}^{2m}$ are not 'too small' as $m \to \infty$, and the calculations on pp. 413–415 of [44] (for the proof of Lemma 2.26 therein), one shows that, for the regular case considered herein (cf. Subsection 2.2), the 'odd' equilibrium measure, $\mu_V^o$ $(\in \mathcal{M}_1(\mathbb{R}))$, is absolutely continuous with respect to Lebesgue measure, that is, the density of the 'odd' equilibrium measure has the representation $\mathrm{d}\mu_V^o(x) := \psi_V^o(x)\,\mathrm{d}x$, $x \in \mathrm{supp}(\mu_V^o)$, where $\psi_V^o(x) \geqslant 0$ on $\overline{J_o}$, with $\psi_V^o(\cdot)$ determined (explicitly) below[12].

Set

$$
\mathcal{H}^o(z) := (\mathcal{F}^o(z))^2 - \int_{J_o} \frac{(4\mathrm{i}(2+\frac{1}{n})^2 \psi_V^o(\xi)(\mathcal{H}\psi_V^o)(\xi) - \frac{4\mathrm{i}}{\pi\xi}(2+\frac{1}{n})\psi_V^o(\xi))}{(\xi - z)} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}}, \quad z \in \mathbb{C} \setminus (J_o \cup \{0\}), \tag{3.1}
$$

where, from the proof of Lemma 3.4,

$$
\mathcal{F}^o(z) = -\frac{1}{\mathrm{i}\pi} \left( \frac{1}{z} + \left(2 + \frac{1}{n}\right) \int_{J_o} \frac{\mathrm{d}\mu_V^o(s)}{s - z} \right), \tag{3.2}
$$

with $\int_{J_o} \frac{\mathrm{d}\mu_V^o(s)}{s-z}$ the Stieltjes transform of the 'odd' equilibrium measure, and

$$
\mathcal{H} \colon \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})} \to \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})},\ f \mapsto (\mathcal{H}f)(z) := \fint_{\mathbb{R}} \frac{f(s)}{z-s} \frac{\mathrm{d}s}{\pi}
$$

denotes the Hilbert transform, with $\fint$ denoting the principle value integral. Via the distributional identities $\frac{1}{x-(x_0 \pm \mathrm{i}0)} = \frac{1}{x-x_0} \pm \mathrm{i}\pi\delta(x-x_0)$, with $\delta(\cdot)$ the Dirac delta function, and $\int_{\xi_1}^{\xi_2} f(\xi)\delta(\xi - x)\mathrm{d}\xi =$

[12] The analysis of [44] is, in some sense, more complicated than the one of the present paper, because, unlike the 'real-line' case considered herein, that is, $\mathrm{supp}(\mu_V^o) =: J_o \subset \overline{\mathbb{R}} \setminus \{0, \pm\infty\}$, the end-point effects at $\pm 1$ in [44] require special consideration (see, also, Section 4 of [44]).

$\begin{cases} f(x), & x\in(\xi_1,\xi_2), \\ 0, & x\in\mathbb{R}\setminus(\xi_1,\xi_2), \end{cases}$ it follows that

$$\mathcal{H}^o_\pm(z)=\begin{cases} (\mathcal{F}^o_\pm(z))^2-⨍_{J_o}\frac{(4\mathrm{i}(2+\frac{1}{n})^2\psi^o_V(\xi)(\mathcal{H}\psi^o_V)(\xi)-\frac{4\mathrm{i}}{\pi\xi}(2+\frac{1}{n})\psi^o_V(\xi))}{(\xi-z)}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\mp\mathfrak{Y}(z), & z\in J_o, \\ (\mathcal{F}^o_\pm(z))^2-\int_{J_o}\frac{(4\mathrm{i}(2+\frac{1}{n})^2\psi^o_V(\xi)(\mathcal{H}\psi^o_V)(\xi)-\frac{4\mathrm{i}}{\pi\xi}(2+\frac{1}{n})\psi^o_V(\xi))}{(\xi-z)}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}, & z\notin J_o, \end{cases}$$

where

$$\mathfrak{Y}(z):=\frac{1}{2}\left(4\mathrm{i}\left(2+\frac{1}{n}\right)^2\psi^o_V(z)(\mathcal{H}\psi^o_V)(z)-\frac{4\mathrm{i}}{\pi z}\left(2+\frac{1}{n}\right)\psi^o_V(z)\right),$$

and $\star^o_\pm(z):=\lim_{\varepsilon\downarrow 0}\star^o(z\pm\mathrm{i}0)$, $\star\in\{\mathcal{H},\mathcal{F}\}$. Recall the definition of $g^o(z)$ given in Lemma 3.4: for $n\in\mathbb{N}$,

$$g^o(z):=\int_{J_o}\ln\left(\frac{(z-s)^{2+\frac{1}{n}}}{zs}\right)\mathrm{d}\mu^o_V(s)=\int_{J_o}\ln\left(\frac{(z-s)^{2+\frac{1}{n}}}{zs}\right)\psi^o_V(s)\,\mathrm{d}s,\quad z\in\mathbb{C}\setminus(-\infty,\max\{0,\max\{J_o\}\});$$

noting the above distributional identities and the fact that $\int_{J_o}\psi^o_V(s)\,\mathrm{d}s=1$, one shows that

$$(g^o_\pm(z))':=\lim_{\varepsilon\downarrow 0}(g^o)'(z\pm\mathrm{i}\varepsilon)=\begin{cases} -\frac{1}{z}-(2+\frac{1}{n})⨍_{J_o}\frac{\psi^o_V(s)}{s-z}\,\mathrm{d}s\mp(2+\frac{1}{n})\pi\mathrm{i}\psi^o_V(z), & z\in J_o, \\ -\frac{1}{z}-(2+\frac{1}{n})\int_{J_o}\frac{\psi^o_V(s)}{s-z}\,\mathrm{d}s, & z\notin J_o, \end{cases}$$

whence one concludes that

$$(g^o_++g^o_-)'(z)=-\frac{2}{z}-2\left(2+\frac{1}{n}\right)⨍_{J_o}\frac{\psi^o_V(s)}{s-z}\,\mathrm{d}s=-\frac{2}{z}+2\left(2+\frac{1}{n}\right)\pi(\mathcal{H}\psi^o_V)(z),\quad z\in J_o,$$

$$(g^o_+-g^o_-)'(z)=\begin{cases} -2\left(2+\frac{1}{n}\right)\pi\mathrm{i}\psi^o_V(z), & z\in J_o, \\ 0, & z\notin J_o. \end{cases}$$

Demanding that (see Lemma 3.6 below) $(g^o_++g^o_-)'(z)=\widetilde{V}'(z)$, $z\in J_o$, one shows from the above that, for $J_o\ni z$, $((g^o(z))'+\frac{1}{z})_++((g^o(z))'+\frac{1}{z})_-=2(2+\frac{1}{n})\pi(\mathcal{H}\psi^o_V)(z)=\frac{2}{z}+\widetilde{V}'(z)\Rightarrow$

$$(\mathcal{H}\psi^o_V)(z)=\frac{1}{2(2+\frac{1}{n})\pi}\left(\frac{2}{z}+\widetilde{V}'(z)\right),\quad z\in J_o.\tag{3.3}$$

From Equation (3.2) and the distributional identities above, one shows that

$$\mathcal{F}^o_\pm(z):=\lim_{\varepsilon\downarrow 0}\mathcal{F}^o(z\pm\mathrm{i}\varepsilon)=\begin{cases} -\frac{1}{\pi\mathrm{i}z}-\mathrm{i}(2+\frac{1}{n})(\mathcal{H}\psi^o_V)(z)\mp(2+\frac{1}{n})\psi^o_V(z), & z\in J_o, \\ -\frac{1}{\pi\mathrm{i}}\left(\frac{1}{z}+(2+\frac{1}{n})\int_{J_o}\frac{\psi^o_V(s)}{s-z}\,\mathrm{d}s\right), & z\notin J_o; \end{cases}\tag{3.4}$$

thus, for $z\in\mathbb{R}\setminus(J_o\cup\{0\})$, $\mathcal{F}^o_+(z)=\mathcal{F}^o_-(z)=-\frac{1}{\pi\mathrm{i}}(\frac{1}{z}+(2+\frac{1}{n})\int_{J_o}\frac{\psi^o_V(s)}{s-z}\,\mathrm{d}s)$. Hence, for $z\notin J_o\cup\{0\}$, one deduces that $\mathcal{H}^o_+(z)=\mathcal{H}^o_-(z)$. For $z\in J_o$, one notes that

$$\mathcal{H}^o_+(z)-\mathcal{H}^o_-(z)=(\mathcal{F}^o_+(z))^2-(\mathcal{F}^o_-(z))^2-4\mathrm{i}\left(2+\frac{1}{n}\right)^2\psi^o_V(z)(\mathcal{H}\psi^o_V)(z)+\frac{4\mathrm{i}}{\pi z}\left(2+\frac{1}{n}\right)\psi^o_V(z),$$

and

$$\begin{aligned}(\mathcal{F}^o_\pm(z))^2 &= -\frac{1}{\pi^2z^2}+\frac{2}{\pi z}\left(2+\frac{1}{n}\right)(\mathcal{H}\psi^o_V)(z)\mp\frac{2\mathrm{i}}{\pi z}\left(2+\frac{1}{n}\right)\psi^o_V(z)-\left(2+\frac{1}{n}\right)^2((\mathcal{H}\psi^o_V)(z))^2 \\ &\pm 2\mathrm{i}\left(2+\frac{1}{n}\right)^2\psi^o_V(z)(\mathcal{H}\psi^o_V)(z)+\left(2+\frac{1}{n}\right)^2(\psi^o_V(z))^2,\end{aligned}$$

whence $(\mathcal{F}^o_+(z))^2-(\mathcal{F}^o_-(z))^2=-\frac{4\mathrm{i}}{\pi z}(2+\frac{1}{n})\psi^o_V(z)+4\mathrm{i}(2+\frac{1}{n})^2\psi^o_V(z)(\mathcal{H}\psi^o_V)(z)\Rightarrow\mathcal{H}^o_+(z)-\mathcal{H}^o_-(z)=0$; thus, for $z\in J_o$, $\mathcal{H}^o_+(z)=\mathcal{H}^o_-(z)$. The above argument shows, therefore, that $\mathcal{H}^o(z)$ is analytic across $\mathbb{R}\setminus\{0\}$; in fact, $\mathcal{H}^o(z)$

is entire for $z \in \mathbb{C}^{*}$. Recalling that $\mu_{V}^{o} \in \mathcal{M}_{1}(\mathbb{R})$, in particular, $\int_{J_o} s^{-m}\, \mathrm{d}\mu_{V}^{o}(s) = \int_{J_o} s^{-m} \psi_{V}^{o}(s)\, \mathrm{d}s < \infty$, $m \in \mathbb{N}$, one shows that, for $|z/s| \ll 1$, with $s \in J_o$ and $z \notin J_o$, via the expansion $\frac{1}{z-s} = -\sum_{k=0}^{l} \frac{z^k}{s^{k+1}} + \frac{z^{l+1}}{s^{l+1}(z-s)}$, $l \in \mathbb{Z}_0^{+}$,

$$(\mathcal{F}^{o}(z))^{2} \underset{z \to 0}{=} -\frac{1}{\pi^{2} z^{2}} - \frac{1}{z}\left(\frac{2}{\pi^{2}}\left(2+\frac{1}{n}\right)\int_{J_o} s^{-1}\, \mathrm{d}\mu_{V}^{o}(s)\right) + \mathcal{O}(1),$$

whence, upon recalling the definition of $\mathcal{H}^{o}(z)$, in particular, for $|z/\xi| \ll 1$, with $\xi \in J_o$ and $z \notin J_o$, via the expansion $\frac{1}{z-\xi} = -\sum_{k=0}^{l} \frac{z^k}{\xi^{k+1}} + \frac{z^{l+1}}{\xi^{l+1}(z-\xi)}$, $l \in \mathbb{Z}_0^{+}$,

$$\int_{J_o} \frac{(4\mathrm{i}(2+\frac{1}{n})^{2} \psi_{V}^{o}(\xi)(\mathcal{H}\psi_{V}^{o})(\xi) - \frac{4\mathrm{i}}{\pi\xi}(2+\frac{1}{n})\psi_{V}^{o}(\xi))}{(\xi - z)} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \underset{z \to 0}{=} \mathcal{O}(1),$$

it follows that

$$\mathcal{H}^{o}(z) \underset{z \to 0}{=} -\frac{1}{\pi^{2} z^{2}} - \frac{1}{z}\left(\frac{2}{\pi^{2}}\left(2+\frac{1}{n}\right)\int_{J_o} s^{-1}\, \mathrm{d}\mu_{V}^{o}(s)\right) + \mathcal{O}(1),$$

which shows that $\mathcal{H}^{o}(z)$ has a pole of order 2 at $z=0$, with $\mathrm{Res}(\mathcal{H}^{o}(z);0) = -\frac{2}{\pi^2}(2+\frac{1}{n}) \int_{J_o} s^{-1}\, \mathrm{d}\mu_{V}^{o}(s)$. One learns from the above analysis that $z^{2}\mathcal{H}^{o}(z)$ is entire: look, in particular, at the behaviour of $z^{2}\mathcal{H}^{o}(z)$ as $|z| \to \infty$. Recalling Equations (3.1) and (3.2), one shows that, for $\mu_{V}^{o} \in \mathcal{M}_{1}(\mathbb{R})$, in particular, $\int_{J_o} \mathrm{d}\mu_{V}^{o}(s) = 1$ and $\int_{J_o} s^{m}\, \mathrm{d}\mu_{V}^{o}(s) < \infty$, $m \in \mathbb{N}$, for $|s/z| \ll 1$, with $s \in J_o$ and $z \notin J_o$, via the expansion $\frac{1}{s-z} = -\sum_{k=0}^{l} \frac{s^k}{z^{k+1}} + \frac{s^{l+1}}{z^{l+1}(s-z)}$, $l \in \mathbb{Z}_0^{+}$,

$$\begin{aligned} &z^{2}\mathcal{H}^{o}(z) + \frac{1}{\pi^{2}}\left(1+\tfrac{1}{n}\right)^{2} - \int_{J_o} s\left(4\mathrm{i}\left(2+\tfrac{1}{n}\right)^{2} \psi_{V}^{o}(s)(\mathcal{H}\psi_{V}^{o})(s) - \tfrac{4\mathrm{i}}{\pi s}\left(2+\tfrac{1}{n}\right)\psi_{V}^{o}(s)\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ &\quad - z\int_{J_o}\left(4\mathrm{i}\left(2+\tfrac{1}{n}\right)^{2} \psi_{V}^{o}(s)(\mathcal{H}\psi_{V}^{o})(s) - \tfrac{4\mathrm{i}}{\pi s}\left(2+\tfrac{1}{n}\right)\psi_{V}^{o}(s)\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{|z| \to \infty}{=} \mathcal{O}(z^{-1}); \end{aligned}$$

thus, due to the entirety of $\mathcal{H}^{o}(z)$, it follows, by a generalisation of Liouville's Theorem, that

$$\begin{aligned} &z^{2}\mathcal{H}^{o}(z) + \frac{1}{\pi^{2}}\left(1+\tfrac{1}{n}\right)^{2} - \int_{J_o} s\left(4\mathrm{i}\left(2+\tfrac{1}{n}\right)^{2} \psi_{V}^{o}(s)(\mathcal{H}\psi_{V}^{o})(s) - \tfrac{4\mathrm{i}}{\pi s}\left(2+\tfrac{1}{n}\right)\psi_{V}^{o}(s)\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\ &\quad - z\int_{J_o}\left(4\mathrm{i}\left(2+\tfrac{1}{n}\right)^{2} \psi_{V}^{o}(s)(\mathcal{H}\psi_{V}^{o})(s) - \tfrac{4\mathrm{i}}{\pi s}\left(2+\tfrac{1}{n}\right)\psi_{V}^{o}(s)\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}} = 0. \end{aligned}$$

Substituting Equation (3.1) into the above formula, one notes that

$$\begin{aligned} &(\mathcal{F}^{o}(z))^{2} - \int_{J_o} \frac{(4\mathrm{i}(2+\frac{1}{n})^{2} \psi_{V}^{o}(\xi)(\mathcal{H}\psi_{V}^{o})(\xi) - \frac{4\mathrm{i}}{\pi\xi}(2+\frac{1}{n})\psi_{V}^{o}(\xi))}{(\xi - z)} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} - \frac{1}{z}\int_{J_o}\left(4\mathrm{i}\left(2+\tfrac{1}{n}\right)^{2} \psi_{V}^{o}(\xi)(\mathcal{H}\psi_{V}^{o})(\xi)\right. \\ &\quad \left.- \tfrac{4\mathrm{i}}{\pi\xi}\left(2+\tfrac{1}{n}\right)\psi_{V}^{o}(\xi)\right)\tfrac{\mathrm{d}\xi}{2\pi\mathrm{i}} + \tfrac{(1+\frac{1}{n})^{2}}{\pi^{2}z^{2}} - \tfrac{1}{z^{2}}\int_{J_o} \xi\left(4\mathrm{i}\left(2+\tfrac{1}{n}\right)^{2} \psi_{V}^{o}(\xi)(\mathcal{H}\psi_{V}^{o})(\xi) - \tfrac{4\mathrm{i}}{\pi\xi}\left(2+\tfrac{1}{n}\right)\psi_{V}^{o}(\xi)\right)\tfrac{\mathrm{d}\xi}{2\pi\mathrm{i}} = 0. \end{aligned}$$

Via Equation (3.3), it follows that $4\mathrm{i}(2+\frac{1}{n})^{2}\psi_{V}^{o}(\xi)(\mathcal{H}\psi_{V}^{o})(\xi) - \frac{4\mathrm{i}}{\pi\xi}(2+\frac{1}{n})\psi_{V}^{o}(\xi) = \frac{2\mathrm{i}}{\pi}(2+\frac{1}{n})\psi_{V}^{o}(\xi)\widetilde{V}'(\xi)$; substituting the latter expression into the above equation, and re-arranging, one obtains,

$$(\mathcal{F}^{o}(z))^{2} - \tfrac{(2+\frac{1}{n})}{\pi^{2}} \int_{J_o} \tfrac{\widetilde{V}'(\xi)\psi_{V}^{o}(\xi)}{\xi - z}\, \mathrm{d}\xi + \tfrac{(1+\frac{1}{n})^{2}}{\pi^{2}z^{2}} - \tfrac{(2+\frac{1}{n})}{\pi^{2}z^{2}} \int_{J_o} \xi \widetilde{V}'(\xi)\psi_{V}^{o}(\xi)\, \mathrm{d}\xi - \tfrac{(2+\frac{1}{n})}{\pi^{2}z} \int_{J_o} \widetilde{V}'(\xi)\psi_{V}^{o}(\xi)\, \mathrm{d}\xi = 0. \tag{3.5}$$

But

$$\begin{aligned} \frac{(2+\frac{1}{n})}{\pi^{2}} \int_{J_o} \frac{\widetilde{V}'(\xi)\psi_{V}^{o}(\xi)}{\xi - z}\, \mathrm{d}\xi &= \frac{(2+\frac{1}{n})}{\pi^{2}} \int_{J_o} \frac{(\widetilde{V}'(\xi) - \widetilde{V}'(z))\psi_{V}^{o}(\xi)}{\xi - z}\, \mathrm{d}\xi + \frac{(2+\frac{1}{n})}{\pi^{2}} \int_{J_o} \frac{\widetilde{V}'(z)\psi_{V}^{o}(\xi)}{\xi - z}\, \mathrm{d}\xi \\ &= \frac{(2+\frac{1}{n})}{\pi^{2}} \int_{J_o} \frac{(\widetilde{V}'(\xi) - \widetilde{V}'(z))\psi_{V}^{o}(\xi)}{\xi - z}\, \mathrm{d}\xi + \frac{\widetilde{V}'(z)}{\pi^{2}} \underbrace{\left(\left(2+\tfrac{1}{n}\right)\int_{J_o} \frac{\psi_{V}^{o}(\xi)}{\xi - z}\, \mathrm{d}\xi\right)}_{= -\mathrm{i}\pi\mathcal{F}^{o}(z) - z^{-1}} \end{aligned}$$

$$=\frac{(2+\frac{1}{n})}{\pi^2}\int_{J_o}\frac{(\widetilde{V}'(\xi)-\widetilde{V}'(z))\psi_V^o(\xi)}{\xi-z}\,\mathrm{d}\xi-\frac{\mathrm{i}\widetilde{V}'(z)\mathcal{F}^o(z)}{\pi}-\frac{\widetilde{V}'(z)}{\pi^2 z}:$$

substituting the above into Equation (3.5), one arrives at, upon completing the square and re-arranging terms,

$$\left(\mathcal{F}^o(z)+\frac{\mathrm{i}\widetilde{V}'(z)}{2\pi}\right)^2+\frac{\mathfrak{q}_V^o(z)}{\pi^2}=0, \tag{3.6}$$

where

$$\mathfrak{q}_V^o(z):=\left(\frac{\widetilde{V}'(z)}{2}\right)^2+\frac{\widetilde{V}'(z)}{z}-\left(2+\frac{1}{n}\right)\int_{J_o}\frac{(\widetilde{V}'(\xi)-\widetilde{V}'(z))\psi_V^o(\xi)}{\xi-z}\,\mathrm{d}\xi$$
$$+\frac{1}{z^2}\left(\left(1+\frac{1}{n}\right)^2-\left(2+\frac{1}{n}\right)\int_{J_o}(\xi+z)\widetilde{V}'(\xi)\psi_V^o(\xi)\,\mathrm{d}\xi\right).$$

(Equation (3.6) above generalises Equation (3.5) for $q^{(0)}(x)$ in [46] for the case when $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is real analytic; moreover, it is analogous to Equation (1.37) of [44].) Note that, since $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfies conditions (2.3)–(2.5), it follows from $\alpha^l-\beta^l=(\alpha-\beta)(\alpha^{l-1}+\alpha^{l-2}\beta+\cdots+\alpha\beta^{l-2}+\beta^{l-1})$, $l\in\mathbb{N}$, that $\mathfrak{q}_V^o(z)$ is real analytic on $J_o$ (and real analytic on $\mathbb{R}\setminus\{0\}$). For $x\in J_o$, set $z:=x+\mathrm{i}\varepsilon$, and consider the $\varepsilon\downarrow 0$ limit of Equation (3.6): $\lim_{\varepsilon\downarrow 0}(\mathcal{F}^o(x+\mathrm{i}\varepsilon)+\frac{\mathrm{i}\widetilde{V}'(x+\mathrm{i}\varepsilon)}{2\pi})^2=(\mathcal{F}_+^o(x)+\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi})^2$ (as $\widetilde{V}$ is real analytic on $J_o$); recalling that $\mathcal{F}_+^o(x)=-\frac{1}{\pi\mathrm{i}x}-\mathrm{i}(2+\frac{1}{n})(\mathcal{H}\psi_V^o)(x)-(2+\frac{1}{n})\psi_V^o(x)$, via Equation (3.3), it follows that $\mathcal{F}_+^o(x)=-\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi}-(2+\frac{1}{n})\psi_V^o(x)\Rightarrow(\mathcal{F}_+^o(x)+\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi})^2=((2+\frac{1}{n})\psi_V^o(x))^2$, whence $(\psi_V^o(x))^2=-\mathfrak{q}_V^o(x)/((2+\frac{1}{n})\pi)^2$, $x\in J_o$, whereupon, using the fact that (see above) $\psi_V^o(x)\geqslant 0\ \forall\ x\in\overline{J_o}$, it follows that $\mathfrak{q}_V^o(x)\leqslant 0$, $x\in J_o$; moreover, as a by-product, decomposing $\mathfrak{q}_V^o(x)$, for $x\in J_o$, into positive and negative parts, that is, $\mathfrak{q}_V^o(x)=(\mathfrak{q}_V^o(x))^+-(\mathfrak{q}_V^o(x))^-$, $x\in J_o$, where $(\mathfrak{q}_V^o(x))^{\pm}:=\max\left\{\pm\mathfrak{q}_V^o(x),0\right\}$ $(\geqslant 0)$, one learns from the above analysis that, for $x\in J_o$, $(\mathfrak{q}_V^o(x))^+\equiv 0$ and $\psi_V^o(x)=\frac{1}{(2+\frac{1}{n})\pi}((\mathfrak{q}_V^o(x))^-)^{1/2}$; and, since $\int_{J_o}\psi_V^o(s)\,\mathrm{d}s=1$, it follows that $\frac{1}{(2+\frac{1}{n})\pi}\int_{J_o}((\mathfrak{q}_V^o(s))^-)^{1/2}\,\mathrm{d}s=1$, which gives rise to the interesting fact that the function $(\mathfrak{q}_V^o(x))^-\not\equiv 0$ on $J_o$. (Even though $(\mathfrak{q}_V^o(x))^-$ depends on $\mathrm{d}\mu_V^o(x)=\psi_V^o(x)\,\mathrm{d}x$, and thus $\psi_V^o(x)=\frac{1}{(2+\frac{1}{n})\pi}((\mathfrak{q}_V^o(x))^-)^{1/2}$ is an implicit representation for $\psi_V^o$, it is still a useful relation which can be used to obtain additional, valuable information about $\psi_V^o$.) For $x\notin J_o$, set $z:=x+\mathrm{i}\varepsilon$, and (again) study the $\varepsilon\downarrow 0$ limit of Equation (3.6): in this case, $\lim_{\varepsilon\downarrow 0}(\mathcal{F}^o(x+\mathrm{i}\varepsilon)+\frac{\mathrm{i}\widetilde{V}'(x+\mathrm{i}\varepsilon)}{2\pi})^2=(\mathcal{F}_+^o(x)+\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi})^2=(\mathcal{F}^o(x)+\frac{\mathrm{i}\widetilde{V}'(x)}{2\pi})^2$; recalling that, for $x\notin J_o$, $\mathcal{F}^o(x)=-\frac{1}{\pi\mathrm{i}}(\frac{1}{x}+(2+\frac{1}{n})\int_{J_o}\frac{\psi_V^o(s)}{s-x}\,\mathrm{d}s)=\frac{\mathrm{i}}{\pi x}-\mathrm{i}(2+\frac{1}{n})(\mathcal{H}\psi_V^o)(x)$, substituting the latter expression into Equation (3.6), one arrives at $(\frac{1}{\pi x}-(2+\frac{1}{n})(\mathcal{H}\psi_V^o)(x)+\frac{\widetilde{V}'(x)}{2\pi})^2=\mathfrak{q}_V^o(x)/\pi^2$, $x\notin J_o$ (since $\widetilde{V}'$ is real analytic on $(\mathbb{R}\setminus\{0\})\setminus J_o$, it follows that $\mathfrak{q}_V^o(x)$, too, is real analytic on $(\mathbb{R}\setminus\{0\})\setminus J_o$, in which case, this latter relation merely states that, for $x=0$, $+\infty=+\infty$), whence $\mathfrak{q}_V^o(x)\geqslant 0\ \forall\ x\notin J_o$.

Now, recalling that, on a compact subset of $\mathbb{R}$, an analytic function changes sign an at most countable number of times, it follows from the above argument, the fact that $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5) is regular (cf. Subsection 2.2), in particular, $\widetilde{V}$ is real analytic in the (open) neighbourhood $\mathbb{U}:=\{z\in\mathbb{C};\ \inf_{q\in J_o}|z-q|<r\in(0,1)\}\setminus\{0\}$, $\mu_V^o$ has compact support, and mimicking a part of the calculations subsumed in the proof of Theorem 1.38 in [44], that $J_o:=\operatorname{supp}(\mu_V^o)=\{x\in\mathbb{R};\ q_V^o(x)\leqslant 0\}$ consists of the disjoint union of a finite number of bounded (real) intervals, with representation $J_o:=\cup_{j=1}^{N+1}J_j^o$, where $J_j^o:=[b_{j-1}^o,a_j^o]$, with $N\in\mathbb{N}$ and finite, $b_0^o:=\min\{J_o\}\notin\{-\infty,0\}$, $a_{N+1}^o:=\max\{J_o\}\notin\{0,+\infty\}$, and $-\infty<b_0^o<a_1^o<b_1^o<a_2^o<\cdots<b_N^o<a_{N+1}^o<+\infty$. (One notes that $\widetilde{V}$ is real analytic in, say, the open neighbourhood $\widetilde{\mathbb{U}}:=\cup_{j=1}^{N+1}\widetilde{\mathbb{U}}_j$, where $\widetilde{\mathbb{U}}_j:=\{z\in\mathbb{C}^*;\ \inf_{q\in J_j^o}|z-q|<r_j\in(0,1)\}$, with $\widetilde{\mathbb{U}}_i\cap\widetilde{\mathbb{U}}_j=\varnothing$, $i\neq j=1,\dots,N+1$.) Furthermore, as a by-product of the above representation for $J_o$, it follows that, since $J_i^o\cap J_j^o=\varnothing$, $i\neq j=1,\dots,N+1$, $\operatorname{meas}(J_o)=\sum_{j=1}^{N+1}|b_{j-1}^o-a_j^o|<+\infty$.

It remains, still, to determine the $2(N+1)$ conditions satisfied by the end-points of the support of the 'odd' equilibrium measure, $\{b_{j-1}^o,a_j^o\}_{j=1}^{N+1}$. Towards this end, one proceeds as follows. From the formula for $\mathcal{F}^o(z)$ given in Equation (3.2):

(i) for $\mu_V^o\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{\mathbb{R}}\mathrm{d}\mu_V^o(s)=1$ and $\int_{\mathbb{R}}s^m\,\mathrm{d}\mu_V^o(s)<\infty$, $m\in\mathbb{N}$, $s\in J_o$ and $z\notin J_o$, with $|s/z|\ll 1$ (e.g., $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$), via the expansion $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, one

gets that $\mathcal{F}^o(z) =_{z\to\infty} \frac{1}{\pi\mathrm{i}}(1+\frac{1}{n})\frac{1}{z}+\mathcal{O}(z^{-2})$;

(ii) for $\mu_V^o \in \mathcal{M}_1(\mathbb{R})$, in particular, $\int_{\mathbb{R}} s^{-m}\,\mathrm{d}\mu_V^o(s) < \infty$, $m\in\mathbb{N}$, $s\in J_o$ and $z\notin J_o$, with $|z/s| \ll 1$ (e.g., $|z| \ll \min_{j=1,\dots,N+1}\{|b_{j-1}^o - a_j^o|\}$), via the expansion $\frac{1}{z-s} = -\sum_{k=0}^{l}\frac{z^k}{s^{k+1}} + \frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, one gets that $\mathcal{F}^o(z) =_{z\to 0} -\frac{1}{\pi\mathrm{i}z}+\mathcal{O}(1)$.

Recalling, also, the formulae for $\mathcal{F}_\pm^o(z)$ given in Equation (3.4), one deduces that $\mathcal{F}_+^o(z)+\mathcal{F}_-^o(z) = -\mathrm{i}\widetilde{V}'(z)/\pi$, $z\in J_o$, and $\mathcal{F}_+^o(z)-\mathcal{F}_-^o(z)=0$, $z\notin J_o$; thus, one learns that $\mathcal{F}^o\colon \mathbb{C}\setminus(J_o\cup\{0\})\to\mathbb{C}$ solves the following (scalar and homogeneous) RHP:

1. $\mathcal{F}^o(z)$ is holomorphic (resp., meromorphic) for $z\in\mathbb{C}\setminus(J_o\cup\{0\})$ (resp., $z\in\mathbb{C}\setminus J_o$);
2. $\mathcal{F}_\pm^o(z) := \lim_{\varepsilon\downarrow 0}\mathcal{F}^o(z\pm\mathrm{i}\varepsilon)$ satisfy the boundary condition $\mathcal{F}_+^o(z)+\mathcal{F}_-^o(z) = -\mathrm{i}\widetilde{V}'(z)/\pi$, $z\in J_o$, with $\mathcal{F}_+^o(z) = \mathcal{F}_-^o(z) := \mathcal{F}^o(z)$ for $z\notin J_o$;
3. $\mathcal{F}^o(z) =_{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}} \frac{1}{\pi\mathrm{i}}(1+\frac{1}{n})\frac{1}{z}+\mathcal{O}(z^{-2})$; and
4. $\mathrm{Res}(\mathcal{F}^o(z);0) = -\frac{1}{\pi\mathrm{i}}$.

The solution of this RHP is (see, for example, [83])

$$\mathcal{F}^o(z) = -\frac{1}{\pi\mathrm{i}z}+(R_o(z))^{1/2}\int_{J_o}\frac{(\frac{2}{\mathrm{i}\pi s}+\frac{\widetilde{V}'(s)}{\mathrm{i}\pi})}{(R_o(s))_+^{1/2}(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}, \quad z\in\mathbb{C}\setminus(J_o\cup\{0\}),$$

where $(R_o(z))^{1/2}$ is defined in the Lemma, with $(R_o(z))_\pm^{1/2} := \lim_{\varepsilon\downarrow 0}(R_o(z\pm\mathrm{i}\varepsilon))^{1/2}$, and the branch of the square root is chosen so that $z^{-(N+1)}(R_o(z))^{1/2} \sim_{\substack{z\to\infty\\ z\in\mathbb{C}_\pm}} \pm 1$. (Note that $(R_o(z))^{1/2}$ is pure imaginary on $J_o$.) It follows from the above integral representation for $\mathcal{F}^o(z)$ that, for $s\in J_o$ and $z\notin J_o$, with $|s/z|\ll 1$ (e.g., $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$), via the expansion $\frac{1}{s-z} = -\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$,

$$\mathcal{F}^o(z) \underset{z\to\infty}{=} -\frac{1}{\mathrm{i}\pi z}+\frac{(z^{N+1}+\cdots)}{z}\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_o(s))_+^{1/2}}\left(1+\frac{s}{z}+\cdots+\frac{s^N}{z^N}+\frac{s^{N+1}}{z^{N+1}}+\cdots\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}}:$$

now, recalling from above that $\mathcal{F}^o(z) =_{\substack{z\to\infty\\ z\in\mathbb{C}_+}} \frac{1}{\pi\mathrm{i}}(1+\frac{1}{n})\frac{1}{z}+\mathcal{O}(z^{-2})$, it follows that, upon removing the secular (growing) terms,

$$\int_{J_o}\left(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi}\right)\frac{s^j}{(R_o(s))_+^{1/2}}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} = 0, \quad j=0,\dots,N$$

(which gives $N+1$ (real) moment conditions), and, upon equating $z^{-1}$ terms,

$$\int_{J_o}\left(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi}\right)\frac{s^{N+1}}{(R_o(s))_+^{1/2}}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} = \frac{1}{\mathrm{i}\pi}\left(2+\frac{1}{n}\right);$$

it remains, thus, to determine an additional $2(N+1)-(N+1)-1=N$ moment conditions. From the integral representation for $\mathcal{F}^o(z)$, a residue calculation shows that

$$\mathcal{F}^o(z) = -\frac{\mathrm{i}\widetilde{V}'(z)}{2\pi}-\frac{(R_o(z))^{1/2}}{2}\oint_{C_{\mathrm{R}}^o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_o(s))^{1/2}(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}, \tag{3.7}$$

where $C_{\mathrm{R}}^o$ $(\subset\mathbb{C}^*)$ denotes the boundary of any open doubly-connected annular region of the type $\{z'\in\mathbb{C};\ 0<r<|z'|<R<+\infty\}$, where the simple outer (resp., inner) boundary $\{z'=R\mathrm{e}^{\mathrm{i}\vartheta},\ 0\leqslant\vartheta\leqslant 2\pi\}$ (resp., $\{z'=r\mathrm{e}^{\mathrm{i}\vartheta},\ 0\leqslant\vartheta\leqslant 2\pi\}$) is traversed clockwise (resp., counter-clockwise), with the numbers $0<r<R<+\infty$ chosen such that, for (any) non-real $z$ in the domain of analyticity of $\widetilde{V}$ (that is, $\mathbb{C}^*$), $\mathrm{int}(C_{\mathrm{R}}^o)\supset J_o\cup\{z\}$. Recall from Equation (3.4) that, for $z\in\mathbb{R}\setminus\overline{J_o}$ $(\supset\cup_{j=1}^N(a_j^o,b_j^o))$, $\mathcal{F}_+^o(z)=\mathcal{F}_-^o(z) = -\frac{1}{\pi\mathrm{i}}(\frac{1}{z}+(2+\frac{1}{n})\int_{J_o}\frac{\psi_V^o(s)}{s-z}\,\mathrm{d}s)$, whence $\mathcal{F}^o(z)+\frac{1}{\pi\mathrm{i}z} = -\mathrm{i}(2+\frac{1}{n})(\mathcal{H}\psi_V^o)(z)$; thus, using Equation (3.7), one arrives at

$$\left(2+\frac{1}{n}\right)(\mathcal{H}\psi_V^o)(z) = \frac{\widetilde{V}'(z)}{2\pi}+\frac{1}{\pi z}+\frac{\mathrm{i}(R_o(z))^{1/2}}{2}\oint_{C_{\mathrm{R}}^o}\frac{(\frac{2}{\pi\mathrm{i}\xi}+\frac{\widetilde{V}'(\xi)}{\pi\mathrm{i}})}{(R_o(\xi))^{1/2}(\xi-z)}\,\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}, \quad z\in\cup_{j=1}^N(a_j^o,b_j^o).$$

A contour integration argument shows that

$$\int_{a_j^o}^{b_j^o}\left((\mathcal{H}\psi_V^o)(s)-\frac{1}{2(2+\frac{1}{n})\pi}\left(\frac{2}{s}+\widetilde{V}'(s)\right)\right)\mathrm{d}s = 0, \quad j=1,\dots,N,$$

whence, using the above expression for $(\mathcal{H}\psi_V^o)(z)$, $z \in \cup_{j=1}^N (a_j^o, b_j^o)$, it follows that

$$\int_{a_j^o}^{b_j^o} \left( \frac{\mathrm{i}(R_o(s))^{1/2}}{2} \oint_{C_{\mathrm{R}}^o} \frac{(\frac{2}{\pi \mathrm{i} \xi} + \frac{\widetilde{V}'(\xi)}{\pi \mathrm{i}})}{(R_o(\xi))^{1/2}(\xi - s)} \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \right) \mathrm{d}s = 0, \quad j = 1, \dots, N: \tag{3.8}$$

now, 'collapsing' the contour $C_{\mathrm{R}}^o$ down to $\mathbb{R} \setminus \{0\}$ and using the Residue Theorem, one shows that

$$\frac{\mathrm{i}(R_o(z))^{1/2}}{2} \oint_{C_{\mathrm{R}}^o} \frac{(\frac{2}{\pi \mathrm{i} \xi} + \frac{\widetilde{V}'(\xi)}{\pi \mathrm{i}})}{(R_o(\xi))^{1/2}(\xi - z)} \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} = -\frac{1}{\pi z} - \frac{\widetilde{V}'(z)}{2\pi} + \mathrm{i}(R_o(z))^{1/2} \int_{J_o} \frac{(\frac{2}{\pi \mathrm{i} \xi} + \frac{\widetilde{V}'(\xi)}{\pi \mathrm{i}})}{(R_o(\xi))_+^{1/2}(\xi - z)} \frac{\mathrm{d}\xi}{2\pi \mathrm{i}};$$

substituting the latter relation into Equation (3.8), one arrives at, after straightforward integration and using the Fundamental Theorem of Calculus, for $j = 1, \dots, N$,

$$\int_{a_j^o}^{b_j^o} \left( \mathrm{i}(R_o(s))^{1/2} \int_{J_o} \left( \frac{\mathrm{i}}{\pi \xi} + \frac{\mathrm{i}\widetilde{V}'(\xi)}{2\pi} \right) \frac{1}{(R_o(\xi))_+^{1/2}(\xi - s)} \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \right) \mathrm{d}s = \frac{1}{2\pi} \ln \left| \frac{a_j^o}{b_j^o} \right| + \frac{1}{4\pi} \left( \widetilde{V}(a_j^o) - \widetilde{V}(b_j^o) \right),$$

which give the remaining $N$ moment conditions determining the end-points of the support of the 'odd' equilibrium measure, $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$. Since $J_o \not\supseteq \{0, \pm\infty\}$ and $\widetilde{V}$ is real analytic on $J_o$,

$$(R_o(s))^{1/2} =_{s \downarrow b_{j-1}^o} \mathcal{O}\left( (s - b_{j-1}^o)^{1/2} \right) \qquad \text{and} \qquad (R_o(s))^{1/2} =_{s \uparrow a_j^o} \mathcal{O}\left( (a_j^o - s)^{1/2} \right), \quad j = 1, \dots, N+1,$$

which shows that all the integrals above constituting the $n$-dependent system of $2(N+1)$ moment conditions for the end-points of the support of $\mu_V^o$ have removable singularities at $b_{j-1}^o, a_j^o$, $j = 1, \dots, N+1$.

Recall from Equation (3.4) that, for $z \in J_o$, $\mathcal{F}_{\pm}^o(z) = -\frac{1}{\pi \mathrm{i} z} - \mathrm{i}(2 + \frac{1}{n})(\mathcal{H}\psi_V^o)(z) \mp (2 + \frac{1}{n})\psi_V^o(z)$: using the fact that, from Equation (3.3), for $z \in J_o$, $(\mathcal{H}\psi_V^o)(z) = \frac{1}{2(2+\frac{1}{n})\pi}(\frac{2}{z} + \widetilde{V}'(z))$, it follows that

$$\mathcal{F}_{\pm}^o(z) = \frac{\widetilde{V}'(z)}{2\pi \mathrm{i}} \mp \left( 2 + \tfrac{1}{n} \right) \psi_V^o(z), \quad z \in J_o.$$

From Equation (3.7), it follows that

$$\mathcal{F}_{\pm}^o(z) = \frac{\widetilde{V}'(z)}{2\pi \mathrm{i}} + \frac{(R_o(z))_{\pm}^{1/2}}{2} \oint_{C_{\mathrm{R}}^o} \frac{(\frac{2}{\pi \mathrm{i} s} + \frac{\widetilde{V}'(s)}{\mathrm{i}\pi})}{(R_o(s))^{1/2}(s - z)} \frac{\mathrm{d}s}{2\pi \mathrm{i}};$$

thus, equating the above two expressions for $\mathcal{F}_{\pm}^o(z)$, one arrives at $\psi_V^o(x) = \frac{1}{2\pi \mathrm{i}}(R_o(x))_+^{1/2} h_V^o(x) \mathbf{1}_{J_o}(x)$, where $h_V^o(z)$ is defined in the Lemma, and $\mathbf{1}_{J_o}(x)$ is the characteristic function of the set $J_o$, which gives rise to the formula for the density of the 'odd' equilibrium measure, $\mathrm{d}\mu_V^o(x) = \psi_V^o(x)\,\mathrm{d}x$ (the integral representation of $h_V^o(z)$ shows that it is analytic in some open subset of $\mathbb{C}^*$ containing $J_o$). Now, recalling that $\widetilde{V}\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying conditions (2.3)–(2.5) is regular, and that, for $s \in J_o$ (resp., $s \in \overline{J_o}$), $\psi_V^o(s) > 0$ (resp., $\psi_V^o(s) \geqslant 0$) and $(R_o(s))_+^{1/2} = \mathrm{i}(|R_o(s)|)^{1/2} \in \mathrm{i}\mathbb{R}_{\pm}$ (resp., $(R_o(s))_+^{1/2} = \mathrm{i}(|R_o(s)|)^{1/2} \in \mathrm{i}\mathbb{R}$), it follows from the formula $\psi_V^o(s) = \frac{1}{2\pi \mathrm{i}}(R_o(s))_+^{1/2} h_V^o(s) \mathbf{1}_{J_o}(s)$ and the regularity assumption, namely, $h_V^o(z) \not\equiv 0$ for $z \in \overline{J_o}$, that $(|R_o(s)|)^{1/2} h_V^o(s) > 0$, $s \in J_o$ (resp., $(|R_o(s)|)^{1/2} h_V^o(s) \geqslant 0$, $s \in \overline{J_o}$).

Finally, it will be shown that, if $\overline{J_o} := \cup_{j=1}^{N+1} [b_{j-1}^o, a_j^o]$, the end-points of the support of the 'odd' equilibrium measure, which satisfy the $n$-dependent system of $2(N+1)$ moment conditions stated in the Lemma, are (real) analytic functions of $z_o$, thus proving the (local) solvability of the $n$-dependent $2(N+1)$ moment conditions. Towards this end, one follows closely the idea of the proof of Theorem 1.3 (iii) in [81] (see, also, Section 8 of [44], and [84]). Recall from Subsection 2.2 that $\widetilde{V}(z) := z_o V(z)$, where $z_o\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}_+$, $(n, \mathscr{N}) \mapsto z_o := \mathscr{N}/n$, and, in the double-scaling limit as $\mathscr{N}, n \to \infty$, $z_o = 1 + o(1)$. Furthermore, from the analysis above, it was shown that the end-points of the support of the 'odd' equilibrium measure were the simple zeros/roots of the function $\mathfrak{q}_V^o(z)$, that is (up to re-arrangement), $\{b_0^o, b_1^o, \dots, b_N^o, a_1^o, a_2^o, \dots, a_{N+1}^o\} = \{x \in \mathbb{R};\ \mathfrak{q}_V^o(x) = 0\}$ (these are the only roots for the regular case studied in this work). The function $\mathfrak{q}_V^o(x) \in \mathbb{R}(x)$ (the algebra of rational functions in $x$ with coefficients in $\mathbb{R}$) is real rational on $\mathbb{R}$ and real analytic on $\mathbb{R} \setminus \{0\}$, it has analytic continuation to $\{z \in \mathbb{C};\ \inf_{p \in \mathbb{R}} |z - p| < r \in (0,1)\} \setminus \{0\}$

(independent of $z_o$), and depends continuously on $z_o$; thus, its simple zeros/roots, that is, $b_{k-1}^o=b_{k-1}^o(z_o)$ and $a_k^o=a_k^o(z_o)$, $k=1,\dots,N+1$, are continuous functions of $z_o$.

Write the large-$z$ (e.g., $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$) asymptotic expansion for $\mathcal{F}^o(z)$ given above as follows:

$$\mathcal{F}^o(z)\underset{z\to\infty}{=}-\frac{1}{\mathrm{i}\pi z}-\frac{(R_o(z))^{1/2}}{2\pi\mathrm{i}z}\sum_{j=0}^{\infty}\mathcal{T}_j^o z^{-j},$$

where

$$\mathcal{T}_j^o:=\int_{J_o}\left(\frac{2}{\mathrm{i}\pi s}+\frac{\widetilde{V}'(s)}{\mathrm{i}\pi}\right)\frac{s^j}{(R_o(s))_+^{1/2}}\,\mathrm{d}s,\quad j\in\mathbb{Z}_0^+.$$

Set, for $n\in\mathbb{N}$,

$$\mathcal{N}_j^o:=\int_{a_j^o}^{b_j^o}\left((\mathcal{H}\psi_V^o)(s)-\frac{1}{2(2+\frac{1}{n})\pi}\left(\frac{2}{s}+\widetilde{V}'(s)\right)\right)\mathrm{d}s,\quad j=1,\dots,N.$$

The ($n$-dependent) $2(N+1)$ moment conditions are, thus, equivalent to $\mathcal{T}_j^o=0$, $j=0,\dots,N$, $\mathcal{T}_{N+1}^o=-2(2+\frac{1}{n})$, and $\mathcal{N}_j^o=0$, $j=1,\dots,N$. It will first be shown that, for regular $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5), the Jacobian of the transformation $\{b_0^o(z_o),\dots,b_N^o(z_o),a_1^o(z_o),\dots,a_{N+1}^o(z_o)\}\mapsto\{\mathcal{T}_0^o,\dots,\mathcal{T}_{N+1}^o,\mathcal{N}_1^o,\dots,\mathcal{N}_N^o\}$, that is, $\mathrm{Jac}(\mathcal{T}_0^o,\dots,\mathcal{T}_{N+1}^o,\mathcal{N}_1^o,\dots,\mathcal{N}_N^o):=\frac{\partial(\mathcal{T}_0^o,\dots,\mathcal{T}_{N+1}^o,\mathcal{N}_1^o,\dots,\mathcal{N}_N^o)}{\partial(b_0^o,\dots,b_N^o,a_1^o,\dots,a_{N+1}^o)}$, is non-zero whenever $b_{j-1}^o=b_{j-1}^o(z_o)$ and $a_j^o=a_j^o(z_o)$, $j=1,\dots,N+1$, are chosen so that $\overline{J_o}=\cup_{j=1}^{N+1}[b_{j-1}^o,a_j^o]$. Using the equation $(\mathcal{H}\psi_V^o)(z)=(2+\frac{1}{n})^{-1}(\mathrm{i}\mathcal{F}^o(z)+\frac{1}{\pi z})$ (cf. Equation (3.2)), one follows the analysis on pp. 778–779 of [81] (see, also, Section 3, Lemma 3.5, of [38]) to show that, for $k=1,\dots,N+1$:

$$\frac{\partial\mathcal{T}_j^o}{\partial b_{k-1}^o}=b_{k-1}^o\frac{\partial\mathcal{T}_{j-1}^o}{\partial b_{k-1}^o}+\frac{1}{2}\mathcal{T}_{j-1}^o,\quad j\in\mathbb{N},\tag{T1}$$

$$\frac{\partial\mathcal{T}_j^o}{\partial a_k^o}=a_k^o\frac{\partial\mathcal{T}_{j-1}^o}{\partial a_k^o}+\frac{1}{2}\mathcal{T}_{j-1}^o,\quad j\in\mathbb{N},\tag{T2}$$

$$\frac{\partial\mathcal{F}^o(z)}{\partial b_{k-1}^o}=-\frac{1}{2\pi\mathrm{i}}\left(\frac{\partial\mathcal{T}_0^o}{\partial b_{k-1}^o}\right)\frac{(R_o(z))^{1/2}}{z-b_{k-1}^o},\quad z\in\mathbb{C}\setminus(\overline{J_o}\cup\{0\}),\tag{F1}$$

$$\frac{\partial\mathcal{F}^o(z)}{\partial a_k^o}=-\frac{1}{2\pi\mathrm{i}}\left(\frac{\partial\mathcal{T}_0^o}{\partial a_k^o}\right)\frac{(R_o(z))^{1/2}}{z-a_k^o},\quad z\in\mathbb{C}\setminus(\overline{J_o}\cup\{0\}),\tag{F2}$$

$$\frac{\partial\mathcal{N}_j^o}{\partial b_{k-1}^o}=-\frac{1}{2(2+\frac{1}{n})\pi}\left(\frac{\partial\mathcal{T}_0^o}{\partial b_{k-1}^o}\right)\int_{a_j^o}^{b_j^o}\frac{(R_o(s))^{1/2}}{s-b_{k-1}^o}\,\mathrm{d}s,\quad j=1,\dots,N,\tag{N1}$$

$$\frac{\partial\mathcal{N}_j^o}{\partial a_k^o}=-\frac{1}{2(2+\frac{1}{n})\pi}\left(\frac{\partial\mathcal{T}_0^o}{\partial a_k^o}\right)\int_{a_j^o}^{b_j^o}\frac{(R_o(s))^{1/2}}{s-a_k^o}\,\mathrm{d}s,\quad j=1,\dots,N;\tag{N2}$$

moreover, if one evaluates Equations (T1) and (T2) on the solution of the $n$-dependent system of $2(N+1)$ moment conditions, that is, $\mathcal{T}_j^o=0$, $j=0,\dots,N$, $\mathcal{T}_{N+1}^o=-2(2+\frac{1}{n})$, and $\mathcal{N}_i^o=0$, $i=1,\dots,N$, one arrives at

$$\frac{\partial\mathcal{T}_j^o}{\partial b_{k-1}^o}=(b_{k-1}^o)^j\frac{\partial\mathcal{T}_0^o}{\partial b_{k-1}^o},\qquad\frac{\partial\mathcal{T}_j^o}{\partial a_k^o}=(a_k^o)^j\frac{\partial\mathcal{T}_0^o}{\partial a_k^o},\quad j=0,\dots,N+1.\tag{S1}$$

Via Equations (N1), (N2), and (S1), one now computes the Jacobian of the transformation $\{b_0^o(z_o),\dots,b_N^o(z_o),a_1^o(z_o),\dots,a_{N+1}^o(z_o)\}\mapsto\{\mathcal{T}_0^o,\dots,\mathcal{T}_{N+1}^o,\mathcal{N}_1^o,\dots,\mathcal{N}_N^o\}$ on the solution of the $n$-dependent system of $2(N+1)$ moment conditions:

$$\mathrm{Jac}(\mathcal{T}_0^o,\dots,\mathcal{T}_{N+1}^o,\mathcal{N}_1^o,\dots,\mathcal{N}_N^o):=\frac{\partial(\mathcal{T}_0^o,\dots,\mathcal{T}_{N+1}^o,\mathcal{N}_1^o,\dots,\mathcal{N}_N^o)}{\partial(b_0^o,\dots,b_N^o,a_1^o,\dots,a_{N+1}^o)}$$

$$
= \begin{vmatrix}
\frac{\partial \mathcal{T}_0^o}{\partial b_0^o} & \frac{\partial \mathcal{T}_0^o}{\partial b_1^o} & \cdots & \frac{\partial \mathcal{T}_0^o}{\partial b_N^o} & \frac{\partial \mathcal{T}_0^o}{\partial a_1^o} & \frac{\partial \mathcal{T}_0^o}{\partial a_2^o} & \cdots & \frac{\partial \mathcal{T}_0^o}{\partial a_{N+1}^o} \\
\frac{\partial \mathcal{T}_1^o}{\partial b_0^o} & \frac{\partial \mathcal{T}_1^o}{\partial b_1^o} & \cdots & \frac{\partial \mathcal{T}_1^o}{\partial b_N^o} & \frac{\partial \mathcal{T}_1^o}{\partial a_1^o} & \frac{\partial \mathcal{T}_1^o}{\partial a_2^o} & \cdots & \frac{\partial \mathcal{T}_1^o}{\partial a_{N+1}^o} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \mathcal{T}_{N+1}^o}{\partial b_0^o} & \frac{\partial \mathcal{T}_{N+1}^o}{\partial b_1^o} & \cdots & \frac{\partial \mathcal{T}_{N+1}^o}{\partial b_N^o} & \frac{\partial \mathcal{T}_{N+1}^o}{\partial a_1^o} & \frac{\partial \mathcal{T}_{N+1}^o}{\partial a_2^o} & \cdots & \frac{\partial \mathcal{T}_{N+1}^o}{\partial a_{N+1}^o} \\
\frac{\partial \mathcal{N}_1^o}{\partial b_0^o} & \frac{\partial \mathcal{N}_1^o}{\partial b_1^o} & \cdots & \frac{\partial \mathcal{N}_1^o}{\partial b_N^o} & \frac{\partial \mathcal{N}_1^o}{\partial a_1^o} & \frac{\partial \mathcal{N}_1^o}{\partial a_2^o} & \cdots & \frac{\partial \mathcal{N}_1^o}{\partial a_{N+1}^o} \\
\frac{\partial \mathcal{N}_2^o}{\partial b_0^o} & \frac{\partial \mathcal{N}_2^o}{\partial b_1^o} & \cdots & \frac{\partial \mathcal{N}_2^o}{\partial b_N^o} & \frac{\partial \mathcal{N}_2^o}{\partial a_1^o} & \frac{\partial \mathcal{N}_2^o}{\partial a_2^o} & \cdots & \frac{\partial \mathcal{N}_2^o}{\partial a_{N+1}^o} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \mathcal{N}_N^o}{\partial b_0^o} & \frac{\partial \mathcal{N}_N^o}{\partial b_1^o} & \cdots & \frac{\partial \mathcal{N}_N^o}{\partial b_N^o} & \frac{\partial \mathcal{N}_N^o}{\partial a_1^o} & \frac{\partial \mathcal{N}_N^o}{\partial a_2^o} & \cdots & \frac{\partial \mathcal{N}_N^o}{\partial a_{N+1}^o}
\end{vmatrix}
$$

$$
= \frac{(-1)^N}{(2(2+\frac{1}{n})\pi)^N} \left( \prod_{k=1}^{N+1} \frac{\partial \mathcal{T}_0^o}{\partial b_{k-1}^o} \frac{\partial \mathcal{T}_0^o}{\partial a_k^o} \right) \left( \prod_{j=1}^{N} \int_{a_j^o}^{b_j^o} (R_o(s_j))^{1/2} \, \mathrm{d}s_j \right) \Delta_d^o,
$$

where

$$
\Delta_d^o := \begin{vmatrix}
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
b_0^o & b_1^o & \cdots & b_N^o & a_1^o & a_2^o & \cdots & a_{N+1}^o \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(b_0^o)^{N+1} & (b_1^o)^{N+1} & \cdots & (b_N^o)^{N+1} & (a_1^o)^{N+1} & (a_2^o)^{N+1} & \cdots & (a_{N+1}^o)^{N+1} \\
\frac{1}{s_1-b_0^o} & \frac{1}{s_1-b_1^o} & \cdots & \frac{1}{s_1-b_N^o} & \frac{1}{s_1-a_1^o} & \frac{1}{s_1-a_2^o} & \cdots & \frac{1}{s_1-a_{N+1}^o} \\
\frac{1}{s_2-b_0^o} & \frac{1}{s_2-b_1^o} & \cdots & \frac{1}{s_2-b_N^o} & \frac{1}{s_2-a_1^o} & \frac{1}{s_2-a_2^o} & \cdots & \frac{1}{s_2-a_{N+1}^o} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{s_N-b_0^o} & \frac{1}{s_N-b_1^o} & \cdots & \frac{1}{s_N-b_N^o} & \frac{1}{s_N-a_1^o} & \frac{1}{s_N-a_2^o} & \cdots & \frac{1}{s_N-a_{N+1}^o}
\end{vmatrix}.
$$

The above determinant, that is, $\Delta_d^o$, has been calculated on pg. 780 of [81] (see, also, Section 5.3, Equations (5.148) and (5.149) of [50]), namely,

$$
\Delta_d^o = \frac{\left( \prod_{j=1}^{N+1} \prod_{k=1}^{N+1} (b_{k-1}^o - a_j^o) \right) \left( \prod_{\substack{j,k=1 \\ j<k}}^{N+1} (a_k^o - a_j^o)(b_{k-1}^o - b_{j-1}^o) \right) \left( \prod_{\substack{j,k=1 \\ j<k}}^{N} (s_k - s_j) \right)}{(-1)^N \prod_{j=1}^{N} \prod_{k=1}^{N+1} (s_j - a_k^o)(s_j - b_{k-1}^o)};
$$

but, for $-\infty < b_0^o < a_1^o < s_1 < b_1^o < a_2^o < s_2 < b_2^o < \cdots < b_{N-1}^o < a_N^o < s_N < b_N^o < a_{N+1}^o < +\infty$, $\Delta_d^o \neq 0$ (which means that it is of a fixed sign), and $\int_{a_j^o}^{b_j^o} (R_o(s_j))^{1/2} \, \mathrm{d}s_j > 0$, $j=1,\ldots,N$, whence

$$
\left( \prod_{j=1}^{N} \int_{a_j^o}^{b_j^o} (R_o(s_j))^{1/2} \, \mathrm{d}s_j \right) \Delta_d^o \neq 0.
$$

It remains to show that $\partial \mathcal{T}_0^o / \partial b_{k-1}^o$ and $\partial \mathcal{T}_0^o / \partial a_k^o$, $k=1,\ldots,N+1$, too, are non-zero; for this purpose, one exploits the fact that $\mathcal{T}_0^o = (\mathrm{i}\pi)^{-1} \int_{J_o} (2s^{-1} + \widetilde{V}'(s))(R_o(s))_+^{-1/2} \, \mathrm{d}s$ is independent of $z$. It follows from Equation (3.7), the integral representation for $h_V^o(z)$ given in the Lemma, and Equations (F1) and (F2) that

$$
\begin{gathered}
\frac{(z-b_{k-1}^o)}{\sqrt{R_o(z)}} \frac{\partial \mathfrak{F}^o(z)}{\partial b_{k-1}^o} = -\frac{1}{2\pi \mathrm{i}} \left( 2 + \frac{1}{n} \right) \left( (z-b_{k-1}^o) \frac{\partial h_V^o(z)}{\partial b_{k-1}^o} - \frac{1}{2} h_V^o(z) \right), \quad k=1,\ldots,N+1, \\
\frac{(z-a_k^o)}{\sqrt{R_o(z)}} \frac{\partial \mathfrak{F}^o(z)}{\partial a_k^o} = -\frac{1}{2\pi \mathrm{i}} \left( 2 + \frac{1}{n} \right) \left( (z-a_k^o) \frac{\partial h_V^o(z)}{\partial a_k^o} - \frac{1}{2} h_V^o(z) \right), \quad k=1,\ldots,N+1:
\end{gathered}
$$

using, now, the $z$-independence of $\mathcal{T}_0^o$, and the fact that, for the case of regular $\widetilde{V} \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying conditions (2.3)–(2.5), $h_V^o(b_{j-1}^o), h_V^o(a_j^o) \neq 0$, $j=1,\ldots,N+1$, one shows that

$$
\left. \frac{(z-b_{k-1}^o)}{\sqrt{R_o(z)}} \frac{\partial \mathfrak{F}^o(z)}{\partial b_{k-1}^o} \right|_{z=b_{k-1}^o} = \frac{1}{4\pi \mathrm{i}} \left( 2 + \frac{1}{n} \right) h_V^o(b_{k-1}^o) \neq 0, \quad k=1,\ldots,N+1,
$$

$$\frac{(z-a_k^o)}{\sqrt{R_o(z)}}\frac{\partial \mathcal{F}^o(z)}{\partial a_k^o}\bigg\vert_{z=a_k^o}=\frac{1}{4\pi\mathrm{i}}\left(2+\frac{1}{n}\right)h_V^o(a_k^o)\neq 0, \quad k=1,\dots,N+1;$$

thus, via Equations (F1) and (F2), one arrives at

$$\frac{\partial \mathcal{T}_0^o}{\partial b_{k-1}^o}=-\frac{1}{2}\left(2+\frac{1}{n}\right)h_V^o(b_{k-1}^o)\neq 0 \qquad \text{and} \qquad \frac{\partial \mathcal{T}_0^o}{\partial a_k^o}=-\frac{1}{2}\left(2+\frac{1}{n}\right)h_V^o(a_k^o)\neq 0, \quad k=1,\dots,N+1,$$

whence

$$\prod_{k=1}^{N+1}\frac{\partial \mathcal{T}_0^o}{\partial b_{k-1}^o}\frac{\partial \mathcal{T}_0^o}{\partial a_k^o}=\left(\frac{1}{2}\left(2+\frac{1}{n}\right)\right)^{2(N+1)}\prod_{k=1}^{N+1}h_V^o(b_{k-1}^o)h_V^o(a_k^o)\neq 0.$$

Hence, $\mathrm{Jac}(\mathcal{T}_0^o,\dots,\mathcal{T}_{N+1}^o,\mathcal{N}_1^o,\dots,\mathcal{N}_N^o)\neq 0$.

It remains, still, to show that $\mathcal{T}_j^o$, $j=0,\dots,N+1$, and $\mathcal{N}_i^o$, $i=1,\dots,N$, are (real) analytic functions of $\{b_{j-1}^o,a_j^o\}_{j=1}^{N+1}$. From the definition of $\mathcal{T}_j^o$, $j\in\mathbb{Z}_0^+$, above, using the fact that they are independent of $z$, thus giving rise to zero residue contributions, a straightforward residue calculus calculation shows that, equivalently,

$$\mathcal{T}_j^o=\frac{1}{2}\oint_{C_{\mathrm{R}}^o}\left(\frac{2}{\mathrm{i}\pi s}+\frac{\widetilde{V}'(s)}{\mathrm{i}\pi}\right)\frac{s^j}{(R_o(s))^{1/2}}\,\mathrm{d}s, \quad j\in\mathbb{Z}_0^+,$$

where (the closed contour) $C_{\mathrm{R}}^o$ has been defined above: the only factor depending on $\{b_{k-1}^o,a_k^o\}_{k=1}^{N+1}$ is $\sqrt{R_o(z)}$. As $\sqrt{R_o(z)}$ is analytic $\forall\ z\in\mathbb{C}\setminus\cup_{j=1}^{N+1}[b_{j-1}^o,a_j^o]$, and since $C_{\mathrm{R}}^o\subset\mathbb{C}\setminus\cup_{j=1}^{N+1}[b_{j-1}^o,a_j^o]$, with $\mathrm{int}(C_{\mathrm{R}}^o)\supset\overline{J_o}\cup\{z\}$, it follows that, in particular, $\sqrt{R_o(z)}\restriction_{C_{\mathrm{R}}^o}$ is an analytic function of $\{b_{j-1}^o,a_j^o\}_{j=1}^{N+1}$, which implies, via the above (equivalent) contour integral representation of $\mathcal{T}_j^o$, $j\in\mathbb{Z}_0^+$, that $\mathcal{T}_k^o$, $k=0,\dots,N+1$, are (real) analytic functions of $\{b_{j-1}^o,a_j^o\}_{j=1}^{N+1}$. Recalling that $(\mathcal{H}\psi_V^o)(z)=(2(2+\frac{1}{n})\pi)^{-1}(2z^{-1}+\widetilde{V}'(z))-\frac{1}{2\pi}(R_o(z))^{1/2}h_V^o(z)$, it follows from the definition of $\mathcal{N}_j^o$, $j=1,\dots,N$, that

$$\mathcal{N}_j^o=-\frac{1}{2\pi}\int_{a_j^o}^{b_j^o}(R_o(s))^{1/2}h_V^o(s)\,\mathrm{d}s, \quad j=1,\dots,N:$$

making the linear change of variables $u_j\colon\mathbb{C}\to\mathbb{C}$, $s\mapsto u_j(s):=(b_j^o-a_j^o)^{-1}(s-a_j^o)$, $j=1,\dots,N$, which take each of the (compact) intervals $[a_j^o,b_j^o]$, $j=1,\dots,N$, onto $[0,1]$, and setting

$$\sqrt{\widehat{R}_o(z)}:=\left(\prod_{k_1=1}^{j}(z-b_{k_1-1}^o)\prod_{k_2=1}^{j-1}(z-a_{k_2}^o)\prod_{k_3=j+1}^{N+1}(a_{k_3}^o-z)\prod_{k_4=j+2}^{N+1}(b_{k_4-1}^o-z)\right)^{1/2},$$

one arrives at

$$\mathcal{N}_j^o=-\frac{1}{2\pi}\left(b_j^o-a_j^o\right)^2\int_0^1\left(u_j(1-u_j)\right)^{1/2}\left(\widehat{R}_o((b_j^o-a_j^o)u_j+a_j^o)\right)^{1/2}h_V^o((b_j^o-a_j^o)u_j+a_j^o)\,\mathrm{d}u_j, \quad j=1,\dots,N.$$

Recalling that $h_V^o(z)$ is analytic on $\mathbb{R}\setminus\{0\}$, in particular, $h_V^o(b_{j-1}^o),h_V^o(a_j^o)\neq 0$, $j=1,\dots,N+1$, and that it is an analytic function of $\{b_{k-1}^o(z_o),a_k^o(z_o)\}_{k=1}^{N+1}$ (since $-\infty<b_0^o<a_1^o<b_1^o<a_2^o<\dots<b_N^o<a_{N+1}^o<+\infty$), and noting from the definition of $\sqrt{\widehat{R}_o(z)}$ above that, it, too, is an analytic function of $(b_{j-1}^o-a_j^o)u_j+a_j^o$, $(j,u_j)\in\{1,\dots,N\}\times[0,1]$, and thus an analytic function of $\{b_{j-1}^o(z_o),a_j^o(z_o)\}_{j=1}^{N+1}$, it follows that $\mathcal{N}_j^o$, $j=1,\dots,N$, are (real) analytic functions of $\{b_{j-1}^o(z_o),a_j^o(z_o)\}_{j=1}^{N+1}$.

Thus, as the Jacobian of the transformation $\{b_0^o(z_o),\dots,b_N^o(z_o),a_1^o(z_o),\dots,a_{N+1}^o(z_o)\}\mapsto\{\mathcal{T}_0^o,\dots,\mathcal{T}_{N+1}^o,\mathcal{N}_1^o,\dots,\mathcal{N}_N^o\}$ is non-zero whenever $\{b_{j-1}^o(z_o),a_j^o(z_o)\}_{j=1}^{N+1}$, the end-points of the support of the 'odd' equilibrium measure, are chosen so that, for regular $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfying conditions (2.3)–(2.5), $\overline{J_o}=\cup_{j=1}^{N+1}[b_{j-1}^o,a_j^o]$, and $\mathcal{T}_j^o$, $j=0,\dots,N+1$, and $\mathcal{N}_k^o$, $k=1,\dots,N$, are (real) analytic functions of $\{b_{j-1}^o(z_o),a_j^o(z_o)\}_{j=1}^{N+1}$, it follows, via the Implicit Function Theorem, that $b_{j-1}^o(z_o),a_j^o(z_o)$, $j=1,\dots,N+1$, are real analytic functions of $z_o$. $\square$

**Remark 3.2.** It turns out that, for $\widetilde{V}\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ (satisfying conditions (2.3)–(2.5)) of the form

$$\widetilde{V}(z)=\sum_{k=-2m_1}^{2m_2}\widetilde{\varrho}_k z^k,$$

with $\widetilde{\varrho}_k\in\mathbb{R}$, $k=-2m_1,\dots,2m_2$, $m_{1,2}\in\mathbb{N}$, and (since $\widetilde{V}(\pm\infty),\widetilde{V}(0)>0$) $\widetilde{\varrho}_{-2m_1},\widetilde{\varrho}_{2m_2}>0$, the integral for $h_V^o(z)$, that is, $h_V^o(z)=\frac{1}{2}(2+\frac{1}{n})^{-1}\oint_{C_{\mathrm{R}}^o}(R_o(s))^{-1/2}(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})(s-z)^{-1}\,\mathrm{d}s$, can be evaluated explicitly. Let $C_{\mathrm{R}}^o=\widetilde{\Gamma}_\infty^o\cup\widetilde{\Gamma}_0^o$, where $\widetilde{\Gamma}_\infty^o:=\{z'=R\mathrm{e}^{\mathrm{i}\vartheta},\ R>1/\varepsilon,\ \vartheta\in[0,2\pi]\}$ (oriented clockwise), and $\widetilde{\Gamma}_0^o:=\{z'=r\mathrm{e}^{\mathrm{i}\vartheta},\ 0<r<\varepsilon,\ \vartheta\in[0,2\pi]\}$ (oriented counter-clockwise), with $\varepsilon$ some arbitrarily fixed, sufficiently small positive real number chosen such that: (i) $\partial\{z'\in\mathbb{C};\ |z'|=\varepsilon\}\cap\partial\{z'\in\mathbb{C};\ |z'|=1/\varepsilon\}=\varnothing$; (ii) $\{z'\in\mathbb{C};\ |z'|<\varepsilon\}\cap(J_o\cup\{z\})=\varnothing$; (iii) $\{z'\in\mathbb{C};\ |z'|>1/\varepsilon\}\cap(J_o\cup\{z\})=\varnothing$; and (iv) $\{z'\in\mathbb{C};\ \varepsilon<|z'|<1/\varepsilon\}\supset J_o\cup\{z\}$. A tedious, but otherwise straightforward, residue calculus calculation shows that

$$\begin{aligned}
h_V^o(z)={}&\frac{z^{2m_2-N-2}}{(2+\frac{1}{n})}\sum_{j=0}^{2m_2-N-2}\ \sideset{}{'}\sum_{\substack{k_0,\dots,k_N\\ 0\leqslant|k|+|l|\leqslant 2m_2-j-N-2\\ k_i\geqslant 0,\ l_i\geqslant 0,\ i\in\{0,\dots,N\}}}\ \sideset{}{'}\sum_{l_0,\dots,l_N}(2m_2-j)\widetilde{\varrho}_{2m_2-j}\left(\prod_{p=0}^{N}\prod_{j_p=0}^{k_p-1}\left(\frac{1}{2}+j_p\right)\right)\\
&\times\left(\prod_{q=0}^{N}\prod_{\widetilde{m}_q=0}^{l_q-1}\left(\frac{1}{2}+\widetilde{m}_q\right)\right)\frac{\left(\prod_{p'=0}^{N}(b_{p'}^o)^{k_{p'}}\right)\left(\prod_{q'=0}^{N}(a_{q'+1}^o)^{l_{q'}}\right)}{\left(\prod_{l'=0}^{N}k_{l'}!\right)\left(\prod_{\widetilde{m}'=0}^{N}l_{\widetilde{m}'}!\right)}z^{-(j+|k|+|l|)}\\
&+\frac{(-1)^{\mathcal{N}_+}(\prod_{k=1}^{N+1}|b_{k-1}^o a_k^o|)^{-1/2}}{(2+\frac{1}{n})z^{2m_1+1}}\sum_{j=-2m_1+1}^{0}\ \sideset{}{''}\sum_{\substack{k_0,\dots,k_N\\ 0\leqslant|k|+|l|\leqslant 2m_1+j\\ k_i\geqslant 0,\ l_i\geqslant 0,\ i\in\{0,\dots,N\}}}\ \sideset{}{''}\sum_{l_0,\dots,l_N}(-2m_1-j)\widetilde{\varrho}_{-2m_1-j}\\
&\times\left(\prod_{p=0}^{N}\prod_{j_p=0}^{k_p-1}\left(\frac{1}{2}+j_p\right)\right)\left(\prod_{q=0}^{N}\prod_{\widetilde{m}_q=0}^{l_q-1}\left(\frac{1}{2}+\widetilde{m}_q\right)\right)\frac{\left(\prod_{p'=0}^{N}(b_{p'}^o)^{k_{p'}}\right)^{-1}\left(\prod_{q'=0}^{N}(a_{q'+1}^o)^{l_{q'}}\right)^{-1}}{\left(\prod_{l'=0}^{N}k_{l'}!\right)\left(\prod_{\widetilde{m}'=0}^{N}l_{\widetilde{m}'}!\right)}\\
&\times z^{|k|+|l|-j}+\frac{2(-1)^{\mathcal{N}_+}(\prod_{k=1}^{N+1}|b_{k-1}^o a_k^o|)^{-1/2}}{(2+\frac{1}{n})z},
\end{aligned}$$

where $\mathcal{N}_+\in\{0,\dots,N+1\}$ is the number of bands to the right of $z=0$, $|k|:=k_0+k_1+\dots+k_N$ $(\geqslant 0)$, $|l|:=l_0+l_1+\dots+l_N$ $(\geqslant 0)$, and the primes (resp., double primes) on the summations mean that all possible sums over $\{k_l\}_{l=0}^N$ and $\{l_k\}_{k=0}^N$ must be taken for which $0\leqslant k_0+\dots+k_N+l_0+\dots+l_N\leqslant 2m_2-j-N-2$, $j=0,\dots,2m_2-N-2$, $k_i\geqslant 0$, $l_i\geqslant 0$, $i=0,\dots,N$ (resp., $0\leqslant k_0+\dots+k_N+l_0+\dots+l_N\leqslant 2m_1+j$, $j=-2m_1+1,\dots,0$, $k_i\geqslant 0$, $l_i\geqslant 0$, $i=0,\dots,N$). It is important to note that all of the above sums are finite sums: any sums for which the upper limit is less than the lower limit are defined to be zero, and any products in which the upper limit is less than the lower limit are defined to be one; for example, $\sum_{j=0}^{-1}(*):=0$ and $\prod_{j=0}^{-1}(*):=1$.

It is interesting to note that one may derive explicit formulae for the various moments of the 'odd' equilibrium measure, that is, $\int_{\mathbb{R}}s^{\pm m}\,\mathrm{d}\mu_V^o(s)=\int_{J_o}s^{\pm m}\psi_V^o(s)\,\mathrm{d}s$, $m\in\mathbb{N}$, in terms of the external field and the function $(R_o(z))^{1/2}$; without loss of generality, and for demonstrative purposes only, consider, say, the following moments: $\int_{J_o}s^{\pm j}\,\mathrm{d}\mu_V^o(s)$, $j=1,2,3$ (the calculations below straightforwardly generalise to $\int_{J_o}s^{\pm(k+3)}\,\mathrm{d}\mu_V^o(s)$, $k\in\mathbb{N}$). Recall the following formulae for $\mathcal{F}^o(z)$ given in Lemma 3.5:

$$\mathcal{F}^o(z)=-\frac{1}{\pi\mathrm{i}z}-\frac{1}{\pi\mathrm{i}}\left(2+\frac{1}{n}\right)\int_{J_o}\frac{\mathrm{d}\mu_V^o(s)}{s-z},\quad z\in\mathbb{C}\setminus(J_o\cup\{0\}),$$

$$\mathcal{F}^o(z)=-\frac{1}{\pi\mathrm{i}z}-(R_o(z))^{1/2}\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_o(s))_+^{1/2}(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}},\quad z\in\mathbb{C}\setminus(J_o\cup\{0\}).$$

One derives the following asymptotic expansions: (1) for $\mu_V^o\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_o}s^{-m}\,\mathrm{d}\mu_V^o(s)<\infty$, $m\in\mathbb{N}$, $s\in J_o$ and $z\notin J_o$, with $|z/s|\ll 1$ (e.g., $|z|\ll\min_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$), via the expansions $\frac{1}{z-s}=$

$-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, and $\ln(1-*)=-\sum_{k=1}^{\infty}\frac{*^k}{k}$, $|*|\ll 1$,

$$\begin{aligned}\mathcal{F}^o(z) \underset{z\to 0}{=} &-\frac{1}{\pi\mathrm{i}z}-\frac{1}{\pi\mathrm{i}}\left(2+\frac{1}{n}\right)\int_{J_o}s^{-1}\,\mathrm{d}\mu_V^o(s)+z\left(-\frac{1}{\pi\mathrm{i}}\left(2+\frac{1}{n}\right)\int_{J_o}s^{-2}\,\mathrm{d}\mu_V^o(s)\right)\\&+z^2\left(-\frac{1}{\pi\mathrm{i}}\left(2+\frac{1}{n}\right)\int_{J_o}s^{-3}\,\mathrm{d}\mu_V^o(s)\right)+\mathcal{O}\left(z^3\right),\end{aligned}$$

and

$$\mathcal{F}^o(z) \underset{z\to 0}{=} -\frac{1}{\pi\mathrm{i}z}+\gamma_V^o\left(\check{Q}_0^o+z(\check{Q}_1^o-\check{P}_0^o\check{Q}_0^o)+z^2(\check{Q}_2^o-\check{P}_0^o\check{Q}_1^o+\check{P}_1^o\check{Q}_0^o)+\mathcal{O}(z^3)\right),$$

where

$$\gamma_V^o:=(-1)^{\mathcal{N}_+}\left(\prod_{j=1}^{N+1}\left|b_{j-1}^o a_j^o\right|\right)^{1/2}, \qquad \check{P}_0^o:=\frac{1}{2}\sum_{j=1}^{N+1}\left(\frac{1}{b_{j-1}^o}+\frac{1}{a_j^o}\right),$$

$$\check{P}_1^o:=\frac{1}{2}(\check{P}_0^o)^2-\frac{1}{4}\sum_{j=1}^{N+1}\left(\frac{1}{(b_{j-1}^o)^2}+\frac{1}{(a_j^o)^2}\right), \qquad \check{Q}_j^o:=-\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_o(s))_+^{1/2}s^{j+1}}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}, \quad j=0,1,2;$$

and (2) for $\mu_V^o\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_o}\mathrm{d}\mu_V^o(s)=1$ and $\int_{J_o}s^m\,\mathrm{d}\mu_V^o(s)<\infty$, $m\in\mathbb{N}$, $s\in J_o$ and $z\notin J_o$, with $|s/z|\ll 1$ (e.g., $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$), via the expansions $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, and $\ln(1-*)=-\sum_{k=1}^{\infty}\frac{*^k}{k}$, $|*|\ll 1$,

$$\begin{aligned}\mathcal{F}^o(z) \underset{z\to\infty}{=} &\frac{1}{\pi\mathrm{i}}\left(1+\frac{1}{n}\right)\frac{1}{z}+\frac{1}{z^2}\left(\frac{1}{\pi\mathrm{i}}\left(2+\frac{1}{n}\right)\int_{J_o}s\,\mathrm{d}\mu_V^o(s)\right)+\frac{1}{z^3}\left(\frac{1}{\pi\mathrm{i}}\left(2+\frac{1}{n}\right)\int_{J_o}s^2\,\mathrm{d}\mu_V^o(s)\right)\\&+\frac{1}{z^4}\left(\frac{1}{\pi\mathrm{i}}\left(2+\frac{1}{n}\right)\int_{J_o}s^3\,\mathrm{d}\mu_V^o(s)\right)+\mathcal{O}\left(\frac{1}{z^5}\right),\end{aligned}$$

and

$$\mathcal{F}^o(z) \underset{z\to\infty}{=} -\frac{1}{\pi\mathrm{i}z}+z^N\left(1+\frac{\check{\alpha}}{z}+\frac{\widetilde{P}_0^o}{z^2}+\frac{\widetilde{P}_1^o}{z^3}+\cdots\right)\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{(R_o(s))_+^{1/2}}\left(1+\cdots+\frac{s^N}{z^N}+\frac{s^{N+1}}{z^{N+1}}+\cdots\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}},$$

where

$$\check{\alpha}:=-\frac{1}{2}\sum_{j=1}^{N+1}\left(b_{j-1}^o+a_j^o\right), \qquad \widetilde{P}_0^o:=\frac{1}{2}(\check{\alpha})^2-\frac{1}{4}\sum_{j=1}^{N+1}\left((b_{j-1}^o)^2+(a_j^o)^2\right),$$

$$\widetilde{P}_1^o:=-\frac{1}{3!}\sum_{j=1}^{N+1}\left((b_{j-1}^o)^3+(a_j^o)^3\right)+\frac{(\check{\alpha})^3}{3!}-\frac{\check{\alpha}}{4}\sum_{j=1}^{N+1}\left((b_{j-1}^o)^2+(a_j^o)^2\right).$$

Recalling the following ($n$-dependent) $N+2$ moment conditions stated in Lemma 3.5,

$$\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^j}{(R_o(s))_+^{1/2}}\,\mathrm{d}s=0, \quad j=0,\dots,N, \qquad \text{and} \qquad \int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+1}}{(R_o(s))_+^{1/2}}\,\mathrm{d}s=2\left(2+\frac{1}{n}\right),$$

and equating the respective pairs of asymptotic expansions above (as $z\to 0$ and $z\to\infty$) for $\mathcal{F}^o(z)$, one arrives at the following expressions for the first three 'positive' and 'negative' moments of the 'odd' equilibrium measure:

$$\int_{J_o}s\,\mathrm{d}\mu_V^o(s)=\frac{1}{2(2+\frac{1}{n})}\left(\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+2}}{(R_o(s))_+^{1/2}}\,\mathrm{d}s-\left(2+\frac{1}{n}\right)\sum_{j=1}^{N+1}(b_{j-1}^o+a_j^o)\right),$$

$$\int_{J_o}s^2\,\mathrm{d}\mu_V^o(s)=\frac{1}{2(2+\frac{1}{n})}\left(\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+3}}{(R_o(s))_+^{1/2}}\,\mathrm{d}s-\frac{1}{2}\left(\sum_{j=1}^{N+1}(b_{j-1}^o+a_j^o)\right)\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+2}}{(R_o(s))_+^{1/2}}\,\mathrm{d}s\right.$$

$$
\begin{aligned}
&+\frac{1}{2}\left(2+\frac{1}{n}\right)\left(\frac{1}{2}\left(\sum_{j=1}^{N+1}(b_{j-1}^{o}+a_{j}^{o})\right)^{2}-\sum_{j=1}^{N+1}((b_{j-1}^{o})^{2}+(a_{j}^{o})^{2})\right)\Bigg),\\
\int_{J_o}s^{3}\,\mathrm{d}\mu_{V}^{o}(s)&=\frac{1}{2(2+\frac{1}{n})}\left(\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+4}}{(R_o(s))_{+}^{1/2}}\,\mathrm{d}s-\frac{1}{2}\left(\sum_{j=1}^{N+1}(b_{j-1}^{o}+a_{j}^{o})\right)\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+3}}{(R_o(s))_{+}^{1/2}}\,\mathrm{d}s\right.\\
&+\frac{1}{4}\left(\frac{1}{2}\left(\sum_{j=1}^{N+1}(b_{j-1}^{o}+a_{j}^{o})\right)^{2}-\sum_{j=1}^{N+1}((b_{j-1}^{o})^{2}+(a_{j}^{o})^{2})\right)\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})s^{N+2}}{(R_o(s))_{+}^{1/2}}\,\mathrm{d}s\\
&-\frac{1}{4}\left(2+\frac{1}{n}\right)\left(\frac{1}{3!}\left(\sum_{j=1}^{N+1}(b_{j-1}^{o}+a_{j}^{o})\right)^{3}+\frac{4}{3}\sum_{j=1}^{N+1}((b_{j-1}^{o})^{3}+(a_{j}^{o})^{3})-\sum_{j=1}^{N+1}(b_{j-1}^{o}+a_{j}^{o})\right.\\
&\left.\left.\times\sum_{k=1}^{N+1}((b_{k-1}^{o})^{2}+(a_{k}^{o})^{2})\right)\right),\\
\int_{J_o}s^{-1}\,\mathrm{d}\mu_{V}^{o}(s)&=\frac{(-1)^{\mathcal{N}_{+}}\left(\prod_{j=1}^{N+1}|b_{j-1}^{o}a_{j}^{o}|\right)^{1/2}}{2(2+\frac{1}{n})}\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{s(R_o(s))_{+}^{1/2}}\,\mathrm{d}s,\\
\int_{J_o}s^{-2}\,\mathrm{d}\mu_{V}^{o}(s)&=\frac{(-1)^{\mathcal{N}_{+}}\left(\prod_{j=1}^{N+1}|b_{j-1}^{o}a_{j}^{o}|\right)^{1/2}}{2(2+\frac{1}{n})}\left(\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{s^{2}(R_o(s))_{+}^{1/2}}\,\mathrm{d}s-\frac{1}{2}\left(\sum_{j=1}^{N+1}\left(\frac{1}{b_{j-1}^{o}}+\frac{1}{a_{j}^{o}}\right)\right)\right.\\
&\left.\times\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{s(R_o(s))_{+}^{1/2}}\,\mathrm{d}s\right),\\
\int_{J_o}s^{-3}\,\mathrm{d}\mu_{V}^{o}(s)&=\frac{(-1)^{\mathcal{N}_{+}}\left(\prod_{j=1}^{N+1}|b_{j-1}^{o}a_{j}^{o}|\right)^{1/2}}{2(2+\frac{1}{n})}\left(\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{s^{3}(R_o(s))_{+}^{1/2}}\,\mathrm{d}s-\frac{1}{2}\left(\sum_{j=1}^{N+1}\left(\frac{1}{b_{j-1}^{o}}+\frac{1}{a_{j}^{o}}\right)\right)\right.\\
&\times\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{s^{2}(R_o(s))_{+}^{1/2}}\,\mathrm{d}s+\left(\frac{1}{8}\left(\sum_{j=1}^{N+1}\left(\frac{1}{b_{j-1}^{o}}+\frac{1}{a_{j}^{o}}\right)\right)^{2}-\frac{1}{4}\sum_{j=1}^{N+1}\left(\frac{1}{(b_{j-1}^{o})^{2}}+\frac{1}{(a_{j}^{o})^{2}}\right)\right)\\
&\left.\times\int_{J_o}\frac{(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})}{s(R_o(s))_{+}^{1/2}}\,\mathrm{d}s\right).
\end{aligned}
$$

It is important to note that all of the above integrals are real valued (since, for $s\in\overline{J_o}$, $(R_o(s))_{+}^{1/2}=\mathrm{i}(|R_o(s)|)^{1/2}\in\mathrm{i}\mathbb{R}$) and bounded (since, for $j=1,\dots,N+1$, $(R_o(s))^{1/2}=_{s\downarrow b_{j-1}^{o}}\mathcal{O}((s-b_{j-1}^{o})^{1/2})$ and $(R_o(s))^{1/2}=_{s\uparrow a_{j}^{o}}\mathcal{O}((a_{j}^{o}-s)^{1/2})$, that is, there are removable singularities at the end-points of the support of the 'odd' equilibrium measure). ∎

**Lemma 3.6.** *Let the external field $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions* (2.3)–(2.5). *Let the 'odd' equilibrium measure, $\mu_{V}^{o}$, and its support,* $\operatorname{supp}(\mu_{V}^{o})=:J_o$ $(\subset\overline{\mathbb{R}}\setminus\{0,\pm\infty\})$, *be as described in Lemma* 3.5, *and let there exist $\ell_o$ $(\in\mathbb{R})$, the 'odd' variational constant, such that*

$$
\begin{aligned}
&2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|x-s|)\psi_{V}^{o}(s)\,\mathrm{d}s-2\ln|x|-\widetilde{V}(x)-\ell_o-2\left(2+\frac{1}{n}\right)Q_o=0,\quad x\in\overline{J_o},\\
&2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|x-s|)\psi_{V}^{o}(s)\,\mathrm{d}s-2\ln|x|-\widetilde{V}(x)-\ell_o-2\left(2+\frac{1}{n}\right)Q_o\leqslant0,\quad x\in\mathbb{R}\setminus\overline{J_o},
\end{aligned}
\tag{3.9}
$$

*where*

$$
Q_o:=\int_{J_o}\ln(|s|)\psi_{V}^{o}(s)\,\mathrm{d}s,
$$

*and, for $\widetilde{V}$ regular, the inequality in the second of Equations* (3.9) *is strict. Then:*

(1) $g_{+}^{o}(z)+g_{-}^{o}(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-})=0$, $z\in\overline{J_o}$, *where* $g_{\pm}^{o}(z):=\lim_{\varepsilon\downarrow0}g^{o}(z\pm\mathrm{i}\varepsilon)$;

(2) $g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})\leqslant 0$, $z\in\mathbb{R}\setminus\overline{J_o}$, *where equality holds for at most a finite number of points, and, for* $\widetilde{V}$ *regular, the inequality is strict;*
(3) $g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}}\in\mathrm{i}f^{\mathbb{R}}_{g^o}$, $z\in\mathbb{R}$, *where* $f^{\mathbb{R}}_{g^o}\colon\mathbb{R}\to\mathbb{R}$ *is some bounded function, and, in particular,* $g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}}=\mathrm{i}\,\mathrm{const.}$, $z\in\mathbb{R}\setminus\overline{J_o}$, *where* $\mathrm{const.}\in\mathbb{R}$;
(4) $\mathrm{i}(g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})'\geqslant 0$, $z\in J_o$, *and where, for* $\widetilde{V}$ *regular, equality holds for at most a finite number of points.*

*Proof.* Set (cf. Lemma 3.5) $J_o:=\cup_{j=1}^{N+1}J^o_j$, where $J^o_j=(b^o_{j-1},a^o_j)=$ the $j$th 'band', with $N\in\mathbb{N}$ and finite, $b^o_0:=\min\{\mathrm{supp}(\mu^o_V)\}\notin\{-\infty,0\}$, $a^o_{N+1}:=\max\{\mathrm{supp}(\mu^o_V)\}\notin\{0,+\infty\}$, and $-\infty<b^o_0<a^o_1<b^o_1<a^o_2<\cdots<b^o_N<a^o_{N+1}<+\infty$, and $\{b^o_{j-1},a^o_j\}_{j=1}^{N+1}$ satisfy the $n$-dependent and (locally) solvable system of $2(N+1)$ moment conditions given in Lemma 3.5. Consider the following cases: **(1)** $z\in\overline{J^o_j}:=[b^o_{j-1},a^o_j]$, $j=1,\dots,N+1$; **(2)** $z\in(a^o_j,b^o_j)=$ the $j$th 'gap', $j=1,\dots,N$; **(3)** $z\in(a^o_{N+1},+\infty)$; and **(4)** $z\in(-\infty,b^o_0)$.

(**1**) Recall the definition of $g^o(z)$ given in Lemma 3.4, namely, $g^o(z):=\int_{J_o}\ln((z-s)^{2+\frac{1}{n}}(zs)^{-1})\psi^o_V(s)\,\mathrm{d}s$, $z\in\mathbb{C}\setminus(-\infty,\max\{0,a^o_{N+1}\})$, where the representation (cf. Lemma 3.5) $\mathrm{d}\mu^o_V(s)=\psi^o_V(s)\,\mathrm{d}s$, $s\in J_o$, was substituted into the latter. For $z\in\overline{J^o_j}$, $j=1,\dots,N+1$, one shows that

$$g^o_\pm(z)=\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|z-s|)\psi^o_V(s)\,\mathrm{d}s\pm\mathrm{i}\pi\left(2+\frac{1}{n}\right)\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s-\int_{J_o}\ln(|s|)\psi^o_V(s)\,\mathrm{d}s$$
$$-\mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\psi^o_V(s)\,\mathrm{d}s-\begin{cases}\ln|z|, & z>0,\\ \ln|z|\pm\mathrm{i}\pi, & z<0,\end{cases}$$

where $g^o_\pm(z):=\lim_{\varepsilon\downarrow 0}g^o(z\pm\mathrm{i}\varepsilon)$, whence

$$g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}}=2\left(2+\frac{1}{n}\right)\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s-2\left(2+\frac{1}{n}\right)\pi\mathrm{i}\int_{J_o\cap\mathbb{R}_+}\psi^o_V(s)\,\mathrm{d}s$$
$$+\begin{cases}0, & z>0,\\ -2\pi\mathrm{i}, & z<0,\end{cases}$$

which shows that $g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}}\in\mathrm{i}\mathbb{R}$, and $\mathrm{Re}(g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})=0$; moreover, using the Fundamental Theorem of Calculus, one shows that $(g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})'=-2(2+\frac{1}{n})\pi\mathrm{i}\psi^o_V(z)$, whence $\mathrm{i}(g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})'=2(2+\frac{1}{n})\pi\psi^o_V(z)\geqslant 0$, since $\psi^o_V(z)\geqslant 0$ $\forall$ $z\in\overline{J_o}$ ($\supset\overline{J^o_j}$, $j=1,\dots,N+1$). Furthermore, using the first of Equations (3.9), one shows that

$$g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})=2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|z-s|)\psi^o_V(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z)$$
$$-\ell_o-2\left(2+\frac{1}{n}\right)Q_o=0,$$

which gives the formula for the ($n$-dependent) 'odd' variational constant $\ell_o$ ($\in\mathbb{R}$), which is the same [81,85] (see, also, Section 7 of [44]) for each compact interval $\overline{J^o_j}$, $j=1,\dots,N+1$; in particular,

$$\ell_o=\frac{1}{\pi}\left(2+\frac{1}{n}\right)\sum_{j=1}^{N+1}\int_{b^o_{j-1}}^{a^o_j}\ln\left(\left|\left(\tfrac{1}{2}(b^o_N+a^o_{N+1})-s\right)s^{-1}\right|\right)(|R_o(s)|)^{1/2}h^o_V(s)\,\mathrm{d}s-2\ln\left|\tfrac{1}{2}(b^o_N+a^o_{N+1})\right|$$
$$-\widetilde{V}\left(\tfrac{1}{2}(b^o_N+a^o_{N+1})\right),$$

where $(|R_o(s)|)^{1/2}h^o_V(s)\geqslant 0$, $j=1,\dots,N+1$, and where there are no singularities in the integrand, since, for (any) $r>0$, $\lim_{|x|\to 0}|x|^r\ln|x|=0$.

(**2**) For $z\in(a^o_j,b^o_j)$, $j=1,\dots,N$, one shows that

$$g^o_\pm(z)=\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|z-s|)\psi^o_V(s)\,\mathrm{d}s\pm\left(2+\frac{1}{n}\right)\pi\mathrm{i}\sum_{k=j+1}^{N+1}\int_{b^o_{k-1}}^{a^o_k}\psi^o_V(s)\,\mathrm{d}s-\int_{J_o}\ln(|s|)\psi^o_V(s)\,\mathrm{d}s$$

$$-\mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\psi_V^o(s)\,\mathrm{d}s-\begin{cases}\ln|z|, & z>0,\\ \ln|z|\pm\mathrm{i}\pi, & z<0,\end{cases}$$

whence

$$g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^- =2\Big(2+\frac{1}{n}\Big)\pi\mathrm{i}\int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s-2\Big(2+\frac{1}{n}\Big)\pi\mathrm{i}\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s + \begin{cases}0, & z>0,\\ -2\pi\mathrm{i}, & z<0,\end{cases}$$

which shows that $g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^- =\mathrm{i}\,\mathrm{const.}$, with $\mathrm{const.}\in\mathbb{R}$, and $\mathrm{Re}(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^-)=0$; moreover, $\mathrm{i}(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^-)'=0$. One notes from the above formulae for $g_\pm^o(z)$ that

$$g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^-) =2\Big(2+\frac{1}{n}\Big)\int_{J_o}\ln(|z-s|)\psi_V^o(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z) -\ell_o-2\Big(2+\frac{1}{n}\Big)Q_o.$$

Recalling that (cf. Lemma 3.5) $\mathcal{H}\colon\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}\to\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}$, $f\mapsto(\mathcal{H}f)(z):=\fint\frac{f(s)}{z-s}\frac{\mathrm{d}s}{\pi}$, where $\fint$ denotes the principal value integral, one shows that, for $z\in(a_j^o,b_j^o)$, $j=1,\dots,N$,

$$2\Big(2+\frac{1}{n}\Big)\int_{J_o}\ln(|z-s|)\psi_V^o(s)\,\mathrm{d}s=2\Big(2+\frac{1}{n}\Big)\pi\int_{a_j^o}^{z}(\mathcal{H}\psi_V^o)(s)\,\mathrm{d}s+2\Big(2+\frac{1}{n}\Big)\int_{J_o}\ln(|a_j^o-s|)\psi_V^o(s)\,\mathrm{d}s;$$

thus,

$$\begin{aligned}g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^-) &=2(2+\tfrac{1}{n})\pi\int_{a_j^o}^{z}(\mathcal{H}\psi_V^o)(s)\,\mathrm{d}s+2(2+\tfrac{1}{n})\int_{J_o}\ln(|a_j^o-s|)\psi_V^o(s)\,\mathrm{d}s\\ &\quad-2\ln|z|-\widetilde{V}(z)-\ell_o-2(2+\tfrac{1}{n})Q_o\\ &=2(2+\tfrac{1}{n})\int_{J_o}\ln(|a_j^o-s|)\psi_V^o(s)\,\mathrm{d}s+2(2+\tfrac{1}{n})\pi\int_{a_j^o}^{z}(\mathcal{H}\psi_V^o)(s)\,\mathrm{d}s\\ &\quad-\int_{a_j^o}^{z}\widetilde{V}'(s)\,\mathrm{d}s-2\int_{a_j^o}^{z}\frac{1}{s}\,\mathrm{d}s-2\ln|a_j^o|-\widetilde{V}(a_j^o)-\ell_o-2(2+\tfrac{1}{n})Q_o\\ &=\int_{a_j^o}^{z}\Big(2\Big(2+\frac{1}{n}\Big)\pi(\mathcal{H}\psi_V^o)(s)-\widetilde{V}'(s)-\frac{2}{s}\Big)\mathrm{d}s,\end{aligned}$$

since

$$2\Big(2+\frac{1}{n}\Big)\int_{J_o}\ln(|a_j^o-s|)\psi_V^o(s)\,\mathrm{d}s-2\ln|a_j^o|-\widetilde{V}(a_j^o)-\ell_o-2\Big(2+\frac{1}{n}\Big)Q_o=0,$$

whence, for $j=1,\dots,N+1$,

$$g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^-)=\int_{a_j^o}^{z}\Big(2\Big(2+\frac{1}{n}\Big)\pi(\mathcal{H}\psi_V^o)(s)-\widetilde{V}'(s)-\frac{2}{s}\Big)\mathrm{d}s,\quad z\in(a_j^o,b_j^o).$$

It was shown in the proof of Lemma 3.5 that $2(2+\frac{1}{n})\pi(\mathcal{H}\psi_V^o)(s)=\widetilde{V}'(s)+\frac{2}{s}-(2+\frac{1}{n})(R_o(s))^{1/2}h_V^o(s)$, $s\in(a_j^o,b_j^o)$, $j=1,\dots,N$, whence

$$g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^+ +\mathfrak{Q}_{\mathcal{A}}^-)=-\Big(2+\frac{1}{n}\Big)\int_{a_j^o}^{z}(R_o(s))^{1/2}h_V^o(s)\,\mathrm{d}s<0,\quad z\in\cup_{j=1}^N(a_j^o,b_j^o):$$

since $h_V^o(z)$ is real analytic on $\mathbb{R}\setminus\{0\}$ and $(R_o(s))^{1/2}h_V^o(s)>0$ $\forall$ $s\in\cup_{j=1}^N(a_j^o,b_j^o)$, it follows that one has equality only at points $z\in\cup_{j=1}^N(a_j^o,b_j^o)$ for which $h_V^o(z)=0$, which are finitely denumerable. (Note that, for $z\in\cup_{j=1}^N(a_j^o,b_j^o)$, $(R_o(s))_+^{1/2}=(R_o(s))_-^{1/2}=(R_o(s))^{1/2}$.)

(**3**) For $z \in (a_{N+1}^{o}, +\infty)$, one shows that

$$g_{\pm}^{o}(z) = \left(2+\frac{1}{n}\right)\int_{J_o} \ln(|z-s|)\psi_V^o(s)\,\mathrm{d}s - \int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s - \mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\psi_V^o(s)\,\mathrm{d}s - \begin{cases}\ln|z|, & z>0,\\ \ln|z|\pm\mathrm{i}\pi, & z<0,\end{cases}$$

whence

$$g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-} = -2\left(2+\frac{1}{n}\right)\pi\mathrm{i}\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s + \begin{cases}0, & z>0,\\ -2\pi\mathrm{i}, & z<0,\end{cases}$$

which shows that $g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-}$ is pure imaginary, and $\mathrm{i}(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-})'=0$. Also, one shows that

$$\begin{aligned} g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-}) &= 2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|z-s|)\psi_V^o(s)\,\mathrm{d}s - 2\ln|z| - \widetilde{V}(z) \\ &\quad -\ell_o - 2\left(2+\frac{1}{n}\right)Q_o; \end{aligned}$$

and, following the analysis of case (**2**) above, one shows that, for $z\in(a_{N+1}^o,+\infty)$,

$$\begin{aligned} 2\left(2+\frac{1}{n}\right)\int_{J_o}\ln(|z-s|)\psi_V^o(s)\,\mathrm{d}s - 2\ln|z| - \widetilde{V}(z) - \ell_o - 2\left(2+\frac{1}{n}\right)Q_o &= \int_{a_{N+1}^o}^{z}\left(2\left(2+\frac{1}{n}\right)\pi(\mathcal{H}\psi_V^o)(s)\right. \\ &\quad \left. -\ \widetilde{V}'(s)-\frac{2}{s}\right)\mathrm{d}s, \end{aligned}$$

thus, via the relation (cf. case (**2**) above) $2(2+\frac{1}{n})\pi(\mathcal{H}\psi_V^o)(s)=\widetilde{V}'(s)+\frac{2}{s}-(2+\frac{1}{n})(R_o(s))^{1/2}h_V^o(s)$, $s\in(a_{N+1}^o,+\infty)$, one arrives at

$$g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-}) = -\left(2+\frac{1}{n}\right)\int_{a_{N+1}^o}^{z}(R_o(s))^{1/2}h_V^o(s)\,\mathrm{d}s<0, \quad z\in(a_{N+1}^o,+\infty).$$

If: (1) $z\to+\infty$ (e.g., $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^o,a_j^o|\}$), $s\in J_o$, and $|s/z|\ll 1$, from $\mu_V^o\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_o}\mathrm{d}\mu_V^o(s)=1$ and $\int_{J_o}s^m\,\mathrm{d}\mu_V^o(s)<\infty$, $m\in\mathbb{N}$, the formula for $g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-})$ above, and the expansions $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, and $\ln(z-s)=_{|z|\to\infty}\ln(z)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{s}{z})^k$, one shows that

$$g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-}) \underset{z\to+\infty}{=} \left(1+\frac{1}{n}\right)\ln(z^2+1)-\widetilde{V}(z)+\mathcal{O}(1),$$

which, upon recalling that (cf. condition (2.4)) $\lim_{|x|\to\infty}(\widetilde{V}(x)/\ln(x^2+1))=+\infty$, shows that $g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-})<0$; and (2) $|z|\to 0$ (e.g., $|z|\ll\min_{j=1,\dots,N+1}\{|b_{j-1}^o,a_j^o|\}$), $s\in J_o$, and $|z/s|\ll 1$, from $\mu_V^o\in\mathcal{M}_1(\mathbb{R})$, in particular, $\int_{J_o}s^{-m}\,\mathrm{d}\mu_V^o(s)<\infty$, $m\in\mathbb{N}$, the above formula for $g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-})$, and the expansions $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, and $\ln(s-z)=_{|z|\to 0}\ln(s)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{s})^k$, one shows that

$$g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-}) \underset{|z|\to 0}{=} \ln(z^{-2}+1)-\widetilde{V}(z)+\mathcal{O}(1),$$

whereupon, recalling that (cf. condition (2.5)) $\lim_{|x|\to 0}(\widetilde{V}(x)/\ln(x^{-2}+1))=+\infty$, it follows that $g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}_{\mathcal{A}}^{+}+\mathfrak{Q}_{\mathcal{A}}^{-})<0$.

(**4**) For $z\in(-\infty,b_0^o)$, one shows that

$$\begin{aligned} g_{\pm}^o(z) &= \left(2+\frac{1}{n}\right)\int_{J_o}\ln(|z-s|)\psi_V^o(s)\,\mathrm{d}s \pm \left(2+\frac{1}{n}\right)\pi\mathrm{i} - \int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s \\ &\quad -\mathrm{i}\pi\int_{J_o\cap\mathbb{R}_-}\psi_V^o(s)\,\mathrm{d}s - \begin{cases}\ln|z|, & z>0,\\ \ln|z|\pm\mathrm{i}\pi, & z<0,\end{cases} \end{aligned}$$

whence

$$g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}}=2\Big(2+\frac{1}{n}\Big)\pi\mathrm{i}-2\Big(2+\frac{1}{n}\Big)\pi\mathrm{i}\int_{J_o\cap\mathbb{R}_+}\psi^o_V(s)\,\mathrm{d}s+\begin{cases}0, & z>0,\\ -2\pi\mathrm{i}, & z<0,\end{cases}$$

which shows that $g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}}$ is pure imaginary, and $\mathrm{i}(g^o_+(z)-g^o_-(z)-\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})'=0$. Also,

$$\begin{aligned}g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})&=2\Big(2+\frac{1}{n}\Big)\int_{J_o}\ln(|z-s|)\psi^o_V(s)\,\mathrm{d}s-2\ln|z|-\widetilde{V}(z)\\&-\ell_o-2\Big(2+\frac{1}{n}\Big)Q_o:\end{aligned}$$

proceeding as in the asymptotic analysis for case (**3**) above, one arrives at

$$g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})\underset{z\to-\infty}{=}\Big(1+\frac{1}{n}\Big)\ln(z^2+1)-\widetilde{V}(z)+\mathcal{O}(1),$$

and

$$g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})\underset{|z|\to0}{=}\ln(z^{-2}+1)-\widetilde{V}(z)+\mathcal{O}(1),$$

whence, via conditions (2.4) and (2.5), $g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-(\mathfrak{Q}^+_{\mathcal{A}}+\mathfrak{Q}^-_{\mathcal{A}})<0$, $z\in(-\infty,b^o_0)$. □

# 4 The Model RHP and Parametrices

In this section, the (normalised at zero) auxiliary RHP for $\overset{o}{\mathcal{M}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ formulated in Lemma 3.4 is augmented, by means of a sequence of contour deformations and transformations *à la* Deift-Venakides-Zhou [1–3], into simpler, 'model' (matrix) RHPs which, as $n\to\infty$, are solved explicitly (in closed form) in terms of Riemann theta functions (associated with the underlying genus-$N$ hyperelliptic Riemann surface) and Airy functions, and which give rise to the leading ($\mathcal{O}(1)$) terms of asymptotics for $\boldsymbol{\pi}_{2n+1}(z)$, $\xi^{(2n+1)}_{-n-1}$ and $\phi_{2n+1}(z)$ stated, respectively, in Theorems 2.3.1 and 2.3.2, and the asymptotic (as $n\to\infty$) analysis of the parametrices, which are 'approximate' solutions of the RHP for $\overset{o}{\mathcal{M}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ in neighbourhoods of the end-points of the support of the 'odd' equilibrium measure, and which give rise to the $\mathcal{O}(n+1/2)$ (and $\mathcal{O}((n+1/2)^2)$) corrections for $\boldsymbol{\pi}_{2n+1}(z)$, $\xi^{(2n+1)}_{-n-1}$ and $\phi_{2n+1}(z)$ stated, respectively, in Theorems 2.3.1 and 2.3.2, is undertaken.

**Lemma 4.1.** *Let the external field* $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5); *furthermore, let* $\widetilde{V}$ *be regular. Let the 'odd' equilibrium measure,* $\mu^o_V$*, and its support,* $\mathrm{supp}(\mu^o_V)=:J_o=\cup_{j=1}^{N+1}J^o_j:=\cup_{j=1}^{N+1}(b^o_{j-1},a^o_j)$*, be as described in Lemma* 3.5*, and, along with* $\ell_o$ $(\in\mathbb{R})$*, the 'odd' variational constant, satisfy the variational conditions given in Lemma* 3.6*, Equations* (3.9)*; moreover, let conditions* (1)–(4) *stated in Lemma* 3.6 *be valid. Then the* RHP *for* $\overset{o}{\mathcal{M}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ *formulated in Lemma* 3.4 *can be equivalently reformulated as follows:* (1) $\overset{o}{\mathcal{M}}(z)$ *is holomorphic for* $z\in\mathbb{C}\setminus\mathbb{R}$; (2) $\overset{o}{\mathcal{M}}_\pm(z):=\lim_{\substack{z'\to z\\ \pm\mathrm{Im}(z')>0}}\overset{o}{\mathcal{M}}(z')$ *satisfy the boundary condition*

$$\overset{o}{\mathcal{M}}_+(z)=\overset{o}{\mathcal{M}}_-(z)\overset{o}{\upsilon}(z),\quad z\in\mathbb{R},$$

*where, for* $i=1,\dots,N+1$ *and* $j=1,\dots,N$,

$$\overset{o}{\upsilon}(z)=\begin{cases}\begin{pmatrix}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\,\mathrm{e}^{\mathrm{i}\mathfrak{q}_o} & 1\\ 0 & \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\,\mathrm{e}^{-\mathrm{i}\mathfrak{q}_o}\end{pmatrix}, & z\in(b^o_{i-1},a^o_i),\\ \begin{pmatrix}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_{b^o_j}^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\,\mathrm{e}^{\mathrm{i}\mathfrak{q}_o} & \mathrm{e}^{n(g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^+_{\mathcal{A}}-\mathfrak{Q}^-_{\mathcal{A}})}\\ 0 & \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_{b^o_j}^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\,\mathrm{e}^{-\mathrm{i}\mathfrak{q}_o}\end{pmatrix}, & z\in(a^o_j,b^o_j),\\ \begin{pmatrix}\mathrm{e}^{\mathrm{i}\mathfrak{q}_o} & \mathrm{e}^{n(g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^+_{\mathcal{A}}-\mathfrak{Q}^-_{\mathcal{A}})}\\ 0 & \mathrm{e}^{-\mathrm{i}\mathfrak{q}_o}\end{pmatrix}, & z\in\mathfrak{I},\end{cases}$$

*with*

$$\mathfrak{q}_o := 4\pi\left(n+\frac{1}{2}\right)\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s,$$

$\mathfrak{I} := (-\infty, b_0^o)\cup(a_{N+1}^o, +\infty)$, $g^o(z)$ *and* $\mathfrak{Q}_{\mathcal{A}}^{\pm}$ *defined in Lemma* 3.4,

$$\pm\mathrm{Re}\left(\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s\right)>0, \quad z\in\mathbb{C}_\pm\cap(\cup_{j=1}^{N+1}\mathbb{U}_j^o),$$

*where* $\mathbb{U}_j^o := \{z\in\mathbb{C}^*;\ \mathrm{Re}(z)\in(b_{j-1}^o, a_j^o),\ \inf_{q\in J_j^o}|z-q|<r_j\in(0,1)\}$, $j=1,\dots,N+1$, *with* $\mathbb{U}_i^o\cap\mathbb{U}_j^o=\varnothing$, $i\neq j=1,\dots,N+1$, *and* $g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+-\mathfrak{Q}_{\mathcal{A}}^-<0$, $z\in\mathfrak{I}\cup(\cup_{j=1}^{N+1}(a_j^o, b_j^o))$; (3) $\overset{o}{\mathcal{M}}(z) \underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to 0}} \mathrm{I}+O(z)$; *and* (4) $\overset{o}{\mathcal{M}}(z) \underset{z\in\mathbb{C}\setminus\mathbb{R}}{=_{z\to\infty}} O(1)$.

*Proof.* Item (1) stated in the Lemma is simply a re-statement of item (1) of Lemma 3.4. Write $\mathbb{R}=(-\infty, b_0^o)\cup(a_{N+1}^o, +\infty)\cup(\cup_{j=1}^{N+1}J_j^o)\cup(\cup_{k=1}^{N}(a_k^o, b_k^o))\cup(\cup_{l=1}^{N+1}\{b_{l-1}^o, a_l^o\})$, where $J_j^o := (b_{j-1}^o, a_j^o)$, $j=1,\dots,N+1$. Recall from the proof of Lemma 3.6 that, for $\widetilde{V}$, $\mu_V^o$, and $\ell_o$ described therein (and in the Lemma): (1) $g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ + \mathfrak{Q}_{\mathcal{A}}^- =$

$$\begin{cases}
2(2+\frac{1}{n})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s - 2(2+\frac{1}{n})\pi\mathrm{i}\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s + \begin{cases}0, & z\in\mathbb{R}_+\cap\overline{J_j^o}, \quad j=1,\dots,N+1,\\ -2\pi\mathrm{i}, & z\in\mathbb{R}_-\cap\overline{J_j^o}, \quad j=1,\dots,N+1,\end{cases}\\
2(2+\frac{1}{n})\pi\mathrm{i}\int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s - 2(2+\frac{1}{n})\pi\mathrm{i}\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s + \begin{cases}0, & z\in\mathbb{R}_+\cap(a_j^o, b_j^o), \quad j=1,\dots,N,\\ -2\pi\mathrm{i}, & z\in\mathbb{R}_-\cap(a_j^o, b_j^o), \quad j=1,\dots,N,\end{cases}\\
-2(2+\frac{1}{n})\pi\mathrm{i}\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s + \begin{cases}0, & z\in\mathbb{R}_+\cap(a_{N+1}^o, +\infty),\\ -2\pi\mathrm{i}, & z\in\mathbb{R}_-\cap(a_{N+1}^o, +\infty),\end{cases}\\
2(2+\frac{1}{n})\pi\mathrm{i} - 2(2+\frac{1}{n})\pi\mathrm{i}\int_{J_o\cap\mathbb{R}_+}\psi_V^o(s)\,\mathrm{d}s + \begin{cases}0, & z\in\mathbb{R}_+\cap(-\infty, b_0^o),\\ -2\pi\mathrm{i}, & z\in\mathbb{R}_-\cap(-\infty, b_0^o),\end{cases}
\end{cases}$$

where $\overline{J_j^o}\ (:= J_j^o\cup\partial J_j^o) = [b_{j-1}^o, a_j^o]$, $j=1,\dots,N+1$; and (2) for $j=1,\dots,N+1$,

$$g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+-\mathfrak{Q}_{\mathcal{A}}^- = \begin{cases}
0, & z\in\cup_{j=1}^{N+1}\overline{J_j^o},\\
-(2+\frac{1}{n})\int_{a_j^o}^{z}(R_o(s))^{1/2}h_V^o(s)\,\mathrm{d}s<0, & z\in(a_j^o, b_j^o),\\
-(2+\frac{1}{n})\int_{a_{N+1}^o}^{z}(R_o(s))^{1/2}h_V^o(s)\,\mathrm{d}s<0, & z\in(a_{N+1}^o, +\infty),\\
(2+\frac{1}{n})\int_z^{b_0^o}(R_o(s))^{1/2}h_V^o(s)\,\mathrm{d}s<0, & z\in(-\infty, b_0^o).
\end{cases}$$

Recall, also, the formula for the 'jump matrix' given in Lemma 3.4, namely,

$$\begin{pmatrix}
\mathrm{e}^{-n(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ + \mathfrak{Q}_{\mathcal{A}}^-)} & \mathrm{e}^{n(g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}_{\mathcal{A}}^+ - \mathfrak{Q}_{\mathcal{A}}^-)}\\
0 & \mathrm{e}^{n(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ + \mathfrak{Q}_{\mathcal{A}}^-)}
\end{pmatrix}.$$

Partitioning $\mathbb{R}$ as given above, one obtains the formula for $\overset{o}{v}(z)$ stated in the Lemma, thus item (2); moreover, items (3) and (4) are re-statements of the respective items of Lemma 3.4. It remains, therefore, to show that $\mathrm{Re}(\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s)$ satisfies the inequalities stated in the Lemma. Recall from the proof of Lemma 3.4 that $g^o(z)$ is uniformly Lipschitz continuous in $\mathbb{C}_\pm$; moreover, via the Cauchy-Riemann conditions, item (4) of Lemma 3.6, that is, $\mathrm{i}(g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ + \mathfrak{Q}_{\mathcal{A}}^-)'\geqslant 0$, $z\in J_o$, implies that the quantity $g_+^o(z)-g_-^o(z)-\mathfrak{Q}_{\mathcal{A}}^+ + \mathfrak{Q}_{\mathcal{A}}^-$ has an analytic continuation, $\mathcal{G}^o(z)$, say, to an open neighbourhood, $\mathbb{U}_V^o$, say, of $J_o=\cup_{j=1}^{N+1}(b_{j-1}^o, a_j^o)$, where $\mathbb{U}_V^o := \cup_{j=1}^{N+1}\mathbb{U}_j^o$, with $\mathbb{U}_j^o := \{z\in\mathbb{C}^*;\ \mathrm{Re}(z)\in(b_{j-1}^o, a_j^o),\ \inf_{q\in J_j^o}|z-q|<r_j\in(0,1)\}$, $j=1,\dots,N+1$, and $\mathbb{U}_i^o\cap\mathbb{U}_j^o=\varnothing$, $i\neq j=1,\dots,N+1$, with the property that $\pm\mathrm{Re}(\mathcal{G}^o(z))>0$, $z\in\mathbb{C}_\pm\cap\mathbb{U}_V^o$. □

**Remark 4.1.** Recalling that the external field $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is regular, that is, $h_V^o(z)\not\equiv 0\ \forall\ z\in\overline{J_j^o} := \cup_{j=1}^{N+1}[b_{j-1}^o, a_j^o]$, the second inequality in Equations (3.9) is strict, namely, $2(2+\frac{1}{n})\int_{J_o}\ln(|x-s|)\psi_V^o(s)\,\mathrm{d}s-$

$2\ln|x|-\widetilde{V}(z)-\ell_o-2(2+\frac{1}{n})Q_o<0$, $x\in\mathbb{R}\setminus\overline{J_o}$, and (from the proof of Lemma 4.1) that $g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-2(2+\frac{1}{n})Q_o<0$, $z\in(-\infty,b^o_0)\cup(a^o_{N+1},+\infty)\cup(\cup_{j=1}^N(a^o_j,b^o_j))$, it follows that

$$\overset{o}{v}(z)\underset{n\to\infty}{=}\begin{cases}\mathrm{e}^{-(4(n+\frac{1}{2})\pi\mathrm{i}\int_{b^o_j}^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s)\sigma_3}\mathrm{e}^{\mathrm{i}\mathfrak{q}_o\sigma_3}(\mathrm{I}+o(1)\sigma_+), & z\in(a^o_j,b^o_j),\quad j=1,\dots,N,\\ \mathrm{e}^{\mathrm{i}\mathfrak{q}_o\sigma_3}(\mathrm{I}+o(1)\sigma_+), & z\in(-\infty,b^o_0)\cup(a^o_{N+1},+\infty),\end{cases}$$

where $o(1)$ denotes terms that are exponentially small. ■

**Proposition 4.1.** *Let the external field $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions* (2.3)–(2.5)*; furthermore, let $\widetilde{V}$ be regular. Let the 'odd' equilibrium measure, $\mu^o_V$, and its support,* $\mathrm{supp}(\mu^o_V)=:J_o=\cup_{j=1}^{N+1}J^o_j:=\cup_{j=1}^{N+1}(b^o_{j-1},a^o_j)$, *be as described in Lemma* 3.5*, and, along with $\ell_o$ $(\in\mathbb{R})$, the 'odd' variational constant, satisfy the variational conditions given in Lemma* 3.6*, Equations* (3.9)*; moreover, let conditions* (1)–(4) *stated in Lemma* 3.6 *be valid. Let $\overset{o}{\mathcal{M}}(z)\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ solve the* RHP *formulated in Lemma* 4.1*. Set*

$$\overset{o}{\mathcal{M}}{}^{\flat}(z)=\begin{cases}\overset{o}{\mathcal{M}}(z)\mathbb{E}^{-\sigma_3}, & z\in\mathbb{C}_+,\\ \overset{o}{\mathcal{M}}(z)\mathbb{E}^{\sigma_3}, & z\in\mathbb{C}_-,\end{cases}$$

*where*

$$\mathbb{E}:=\exp(\mathrm{i}\mathfrak{q}_o/2)=\exp\left(\mathrm{i}2\pi\left(n+\frac{1}{2}\right)\int_{J_o\cap\mathbb{R}_+}\psi^o_V(s)\,\mathrm{d}s\right).$$

*Then $\overset{o}{\mathcal{M}}{}^{\flat}(z)\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ solves the following* RHP: (1) *$\overset{o}{\mathcal{M}}{}^{\flat}(z)$ is holomorphic for $z\in\mathbb{C}\setminus\mathbb{R}$;* (2) $\overset{o}{\mathcal{M}}{}^{\flat}_{\pm}(z):=\lim_{\substack{z'\to z\\ \pm\mathrm{Im}(z')>0}}\overset{o}{\mathcal{M}}{}^{\flat}(z')$ *satisfy the boundary condition*

$$\overset{o}{\mathcal{M}}{}^{\flat}_+(z)=\overset{o}{\mathcal{M}}{}^{\flat}_-(z)\mathcal{V}_{\overset{o}{\mathcal{M}}{}^{\flat}}(z),\quad z\in\mathbb{R},$$

*where, for $i=1,\dots,N+1$ and $j=1,\dots,N$,*

$$\mathcal{V}_{\overset{o}{\mathcal{M}}{}^{\flat}}(z)=\begin{cases}\begin{pmatrix}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s} & 1\\ 0 & \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\end{pmatrix}, & z\in(b^o_{i-1},a^o_i),\\ \begin{pmatrix}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_{b^o_j}^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s} & \mathrm{e}^{n(g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^+_{\mathcal{A}}-\mathfrak{Q}^-_{\mathcal{A}})}\\ 0 & \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_{b^o_j}^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\end{pmatrix}, & z\in(a^o_j,b^o_j),\\ \mathrm{I}+\mathrm{e}^{n(g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^+_{\mathcal{A}}-\mathfrak{Q}^-_{\mathcal{A}})}\sigma_+, & z\in\mathfrak{I},\end{cases}$$

*with $\mathfrak{I}:=(-\infty,b^o_0)\cup(a^o_{N+1},+\infty)$, $g^o(z)$ and $\mathfrak{Q}^{\pm}_{\mathcal{A}}$ defined in Lemma* 3.4*,*

$$\pm\mathrm{Re}\left(\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s\right)>0,\quad z\in\mathbb{C}_{\pm}\cap(\cup_{j=1}^{N+1}\mathbb{U}^o_j),$$

*where $\mathbb{U}^o_j:=\{z\in\mathbb{C}^*;\,\mathrm{Re}(z)\in(b^o_{j-1},a^o_j),\,\inf_{q\in J^o_j}|z-q|<r_j\in(0,1)\}$, $j=1,\dots,N+1$, with $\mathbb{U}^o_i\cap\mathbb{U}^o_j=\varnothing$, $i\neq j=1,\dots,N+1$, and $g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^+_{\mathcal{A}}-\mathfrak{Q}^-_{\mathcal{A}}<0$, $z\in\mathfrak{I}\cup(\cup_{j=1}^{N+1}(a^o_j,b^o_j))$;* (3) $\overset{o}{\mathcal{M}}{}^{\flat}(z)\underset{\substack{z\to0\\ z\in\mathbb{C}_+}}{=}(\mathrm{I}+\mathcal{O}(z))\mathbb{E}^{-\sigma_3}$ *and* $\overset{o}{\mathcal{M}}{}^{\flat}(z)\underset{\substack{z\to0\\ z\in\mathbb{C}_-}}{=}(\mathrm{I}+\mathcal{O}(z))\mathbb{E}^{\sigma_3}$*; and* (4) $\overset{o}{\mathcal{M}}{}^{\flat}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}\mathcal{O}(1)$.

*Proof.* Follows from the definition of $\overset{o}{\mathcal{M}}{}^{\flat}(z)$ in terms of $\overset{o}{\mathcal{M}}(z)$ given in the Proposition and the RHP for $\overset{o}{\mathcal{M}}(z)$ formulated in Lemma 4.1. □

**Lemma 4.2.** *Let the external field $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ satisfy conditions* (2.3)–(2.5)*; furthermore, let $\widetilde{V}$ be regular. Let the 'odd' equilibrium measure, $\mu^o_V$, and its support,* $\mathrm{supp}(\mu^o_V)=:J_o=\cup_{j=1}^{N+1}J^o_j:=\cup_{j=1}^{N+1}(b^o_{j-1},a^o_j)$, *be as described in Lemma* 3.5*, and, along with $\ell_o$ $(\in\mathbb{R})$, the 'odd' variational constant, satisfy the variational conditions*

*given in Lemma* 3.6, *Equations* (3.9); *moreover, let conditions* (1)–(4) *stated in Lemma* 3.6 *be valid. Let* $\overset{o}{\mathcal{M}}^{\flat}(z)\colon \mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ *solve the* RHP *formulated in Proposition* 4.1, *and let the deformed (and oriented) contour* $\Sigma_o^{\sharp}:=\mathbb{R}\cup(\cup_{j=1}^{N+1}(J_j^{o,\frown}\cup J_j^{o,\smile}))$ *be as in Figure* 8 *below; furthermore,* $\cup_{j=1}^{N+1}(\Omega_j^{o,\frown}\cup\Omega_j^{o,\smile}\cup J_j^{o,\frown}\cup J_j^{o,\smile})\subset\cup_{j=1}^{N+1}\mathbb{U}_j^o$ *(Figure* 8*), where* $\mathbb{U}_j^o$, $j=1,\dots,N+1$, *is defined in Lemma* 4.1. *Set*

$$\overset{o}{\mathcal{M}}^{\sharp}(z):=\begin{cases}\overset{o}{\mathcal{M}}^{\flat}(z), & z\in\mathbb{C}\setminus(\Sigma_o^{\sharp}\cup(\cup_{j=1}^{N+1}(\Omega_j^{o,\frown}\cup\Omega_j^{o,\smile}))),\\ \overset{o}{\mathcal{M}}^{\flat}(z)\left(\mathrm{I}-\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\sigma_-\right), & z\in\mathbb{C}_+\cap(\cup_{j=1}^{N+1}\Omega_j^{o,\frown}),\\ \overset{o}{\mathcal{M}}^{\flat}(z)\left(\mathrm{I}+\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\sigma_-\right), & z\in\mathbb{C}_-\cap(\cup_{j=1}^{N+1}\Omega_j^{o,\smile}).\end{cases}$$

*Then* $\overset{o}{\mathcal{M}}^{\sharp}\colon\mathbb{C}\setminus\Sigma_o^{\sharp}\to\mathrm{SL}_2(\mathbb{C})$ *solves the following, equivalent* RHP: (1) $\overset{o}{\mathcal{M}}^{\sharp}(z)$ *is holomorphic for* $z\in\mathbb{C}\setminus\Sigma_o^{\sharp}$; (2) $\overset{o}{\mathcal{M}}^{\sharp}_{\pm}(z):=\lim_{\substack{z'\to z\\ z'\,\in\,\pm\,\text{side of}\,\Sigma_o^{\sharp}}}\overset{o}{\mathcal{M}}^{\sharp}(z')$ *satisfy the boundary condition*

$$\overset{o}{\mathcal{M}}^{\sharp}_{+}(z)=\overset{o}{\mathcal{M}}^{\sharp}_{-}(z)\overset{o}{\upsilon}^{\sharp}(z),\quad z\in\Sigma_o^{\sharp},$$

*where, for* $i=1,\dots,N+1$ *and* $j=1,\dots,N$,

$$\overset{o}{\upsilon}^{\sharp}(z)=\begin{cases}\mathrm{i}\sigma_2, & z\in J_i^o,\\ \mathrm{I}+\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\sigma_-, & z\in J_i^{o,\frown},\\ \mathrm{I}+\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\sigma_-, & z\in J_i^{o,\smile},\\ \begin{pmatrix}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s} & \mathrm{e}^{n(g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}_{\mathscr{A}}^+-\mathfrak{Q}_{\mathscr{A}}^-)}\\ 0 & \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\end{pmatrix}, & z\in(a_j^o,b_j^o),\\ \mathrm{I}+\mathrm{e}^{n(g_+^o(z)+g_-^o(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}_{\mathscr{A}}^+-\mathfrak{Q}_{\mathscr{A}}^-)}\sigma_+, & z\in\mathfrak{I},\end{cases}$$

*with* $\mathfrak{I}:=(-\infty,b_0^o)\cup(a_{N+1}^o,+\infty)$, *and* $\mathrm{Re}(\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s)>0$ *(resp.*, $\mathrm{Re}(\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s)<0$*)*, $z\in\Omega_j^{o,\frown}$ *(resp.*, $z\in\Omega_j^{o,\smile}$*)*; (3)

$$\overset{o}{\mathcal{M}}^{\sharp}(z)\underset{\substack{z\to 0\\ z\in\mathbb{C}_+\setminus\cup_{j=1}^{N+1}(J_j^{o,\frown}\cup\Omega_j^{o,\frown})}}{=}(\mathrm{I}+\mathcal{O}(z))\mathbb{E}^{-\sigma_3},$$

$$\overset{o}{\mathcal{M}}^{\sharp}(z)\underset{\substack{z\to 0\\ z\in\mathbb{C}_-\setminus\cup_{j=1}^{N+1}(J_j^{o,\smile}\cup\Omega_j^{o,\smile})}}{=}(\mathrm{I}+\mathcal{O}(z))\mathbb{E}^{\sigma_3};$$

*and* (4)

$$\overset{o}{\mathcal{M}}^{\sharp}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus(\Sigma_o^{\sharp}\cup(\cup_{j=1}^{N+1}(\Omega_j^{o,\frown}\cup\Omega_j^{o,\smile})))}}{=}\mathcal{O}(1).$$

*Proof.* Items (1), (3), and (4) in the formulation of the RHP for $\overset{o}{\mathcal{M}}^{\sharp}\colon\mathbb{C}\setminus\Sigma_o^{\sharp}\to\mathrm{SL}_2(\mathbb{C})$ follow from the definition of $\overset{o}{\mathcal{M}}^{\sharp}(z)$ (in terms of $\overset{o}{\mathcal{M}}^{\flat}(z)$) given in the Lemma and the respective items (1), (3), and (4) for the RHP for $\overset{o}{\mathcal{M}}^{\flat}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_2(\mathbb{C})$ stated in Proposition 4.1; it remains, therefore, to verify item (2), that is, the formula for $\overset{o}{\upsilon}^{\sharp}(z)$. Recall from item (2) of Proposition 4.1 that, for $z\in(b_{j-1}^o,a_j^o)$, $j=1,\dots,N+1$, $\overset{o}{\mathcal{M}}^{\flat}_{+}(z)=\overset{o}{\mathcal{M}}^{\flat}_{-}(z)\mathcal{V}_{\overset{o}{\mathcal{M}}^{\flat}}(z)$, where $\mathcal{V}_{\overset{o}{\mathcal{M}}^{\flat}}(z)=\begin{pmatrix}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s} & 1\\ 0 & \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\end{pmatrix}$: noting the matrix factorisation

$$\begin{pmatrix}\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s} & 1\\ 0 & \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s}\end{pmatrix}=\begin{pmatrix}1 & 0\\ \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s} & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

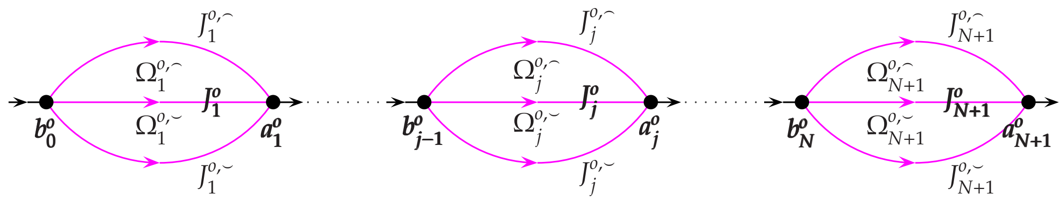


Figure 8: Oriented/deformed contour $\Sigma_o^\sharp := \mathbb{R} \cup (\cup_{j=1}^{N+1}(J_j^{o,\frown} \cup J_j^{o,\smile}))$

$$\times \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-4(n+\frac{1}{2})\pi \mathrm{i} \int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s} & 1 \end{pmatrix},$$

it follows that, $z \in (b_{j-1}^o, a_j^o)$, $j=1,\dots,N+1$,

$$\overset{o}{\mathcal{M}}{}^\flat_+(z) \begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{-4(n+\frac{1}{2})\pi \mathrm{i} \int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s} & 1 \end{pmatrix} = \overset{o}{\mathcal{M}}{}^\flat_-(z) \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{4(n+\frac{1}{2})\pi \mathrm{i} \int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s} & 1 \end{pmatrix} \mathrm{i}\sigma_2.$$

It was shown in Lemma 4.1 that $\pm\mathrm{Re}(\mathrm{i}\int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s) > 0$ for $z \in \mathbb{C}_\pm \cap \mathbb{U}_j^o$, where $\mathbb{U}_j^o := \{z \in \mathbb{C}^*;\ \mathrm{Re}(z) \in (b_{j-1}^o, a_j^o),\ \inf_{q \in J_j^o} |z-q| < r_j \in (0,1)\}$, $j=1,\dots,N+1$, with $\mathbb{U}_i^o \cap \mathbb{U}_j^o = \varnothing$, $i \neq j = 1,\dots,N+1$, and $J_j^o := (b_{j-1}^o, a_j^o)$, $j=1,\dots,N+1$. (One notes that the terms $\pm 4(n+\frac{1}{2})\pi \mathrm{i} \int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s$, which are pure imaginary for $z \in \mathbb{R}$, and corresponding to which $\exp(\pm 4(n+\frac{1}{2})\pi \mathrm{i} \int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s)$ are undulatory, are continued analytically to $\mathbb{C}_\pm \cap (\cup_{j=1}^{N+1} \mathbb{U}_j^o)$, respectively, corresponding to which $\exp(\pm 4(n+\frac{1}{2})\pi \mathrm{i} \int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s)$ are exponentially decreasing as $n \to \infty$). As per the DZ non-linear steepest-descent method [1, 2] (see, also, the extension [3]), one now 'deforms' the original (and oriented) contour $\mathbb{R}$ to the deformed, or extended, (and oriented) contour/skeleton $\Sigma_o^\sharp := \mathbb{R} \cup (\cup_{j=1}^{N+1}(J_j^{o,\frown} \cup J_j^{o,\smile}))$ (Figure 8) in such a way that the upper (resp., lower) 'lips' of the 'lenses' $J_j^{o,\frown}$ (resp., $J_j^{o,\smile}$), $j=1,\dots,N+1$, which are the boundaries of $\Omega_j^{o,\frown}$ (resp., $\Omega_j^{o,\smile}$), $j=1,\dots,N+1$, respectively, lie within the domain of analytic continuation of $g_+^o(z) - g_-^o(z) - \mathfrak{Q}_{\mathcal{A}}^+ + \mathfrak{Q}_{\mathcal{A}}^-$ (cf. the proof of Lemma 4.1), that is, $\cup_{j=1}^{N+1}(\Omega_j^{o,\frown} \cup \Omega_j^{o,\smile} \cup J_j^{o,\frown} \cup J_j^{o,\smile}) \subset \cup_{j=1}^{N+1} \mathbb{U}_j^o$; in particular, each (oriented) interval $J_j^o = (b_{j-1}^o, a_j^o)$, $j=1,\dots,N+1$, in the original (and oriented) contour $\mathbb{R}$ is 'split' (or branched) into three, and the new (and oriented) contour $\Sigma_o^\sharp$ is the old contour ($\mathbb{R}$) together with the (oriented) boundary of $N+1$ lens-shaped regions, one region surrounding each (bounded and oriented) interval $J_j^o$. Now, recalling the definition of $\overset{o}{\mathcal{M}}{}^\sharp(z)$ (in terms of $\overset{o}{\mathcal{M}}{}^\flat(z)$) stated in the Lemma, and the expressions for (the jump matrix) $\mathcal{V}_{\overset{o}{\mathcal{M}}{}^\flat}(z)$ given in Proposition 4.1, one arrives at the formula for $\overset{o}{v}{}^\sharp(z)$ given in item (2) of the Lemma. □

**Remark 4.2.** The jump condition stated in item (2) of Lemma 4.2, that is, $\overset{o}{\mathcal{M}}{}^\sharp_+(z) = \overset{o}{\mathcal{M}}{}^\sharp_-(z)\overset{o}{v}{}^\sharp(z)$, $z \in \Sigma_o^\sharp$, with $\overset{o}{v}{}^\sharp(z)$ given therein, should, of course, be understood as follows: the $\mathrm{SL}_2(\mathbb{C})$-valued functions $\overset{o}{\mathcal{M}}{}^\sharp \restriction_{\mathbb{C}_\pm \setminus \Sigma_o^\sharp}$ have a continuous extension to $\Sigma_o^\sharp$ with boundary values $\overset{o}{\mathcal{M}}{}^\sharp_\pm(z) := \lim_{\substack{z' \to z \in \Sigma_o^\sharp \\ z' \in \pm \text{ side of } \Sigma_o^\sharp}} \overset{o}{\mathcal{M}}{}^\sharp(z')$ satisfying the above jump relation ($\overset{o}{\mathcal{M}}{}^\sharp(z)$ is continuous in each component of $\mathbb{C} \setminus \Sigma_o^\sharp$ up to the boundary with boundary values $\overset{o}{\mathcal{M}}{}^\sharp_\pm(z)$ satisfying the above jump relation on $\Sigma_o^\sharp$). ■

Recalling from Proposition 4.1 that, for $z \in (-\infty, b_0^o) \cup (a_{N+1}^o, +\infty) \cup (\cup_{j=1}^N (a_j^o, b_j^o))$, $g_+^o(z) + g_-^o(z) - \widetilde{V}(z) - \ell_o - \mathfrak{Q}_{\mathcal{A}}^+ - \mathfrak{Q}_{\mathcal{A}}^- < 0$, and, from Lemma 4.2, $\mathrm{Re}(\mathrm{i}\int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s) > 0$ for $z \in J_j^{o,\frown}$ (resp., $\mathrm{Re}(\mathrm{i}\int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s) < 0$ for $z \in J_j^{o,\smile}$), $j=1,\dots,N+1$, one arrives at the following large-$n$ asymptotic behaviour for the jump

matrix $\overset{o}{v}^{\sharp}(z)$: for $i=1,\dots,N+1$ and $j=1,\dots,N$,

$$\overset{o}{v}^{\sharp}(z) \underset{n\to\infty}{=} \begin{cases} \mathrm{i}\sigma_2, & z\in J_i^o,\\ \mathrm{I}+\mathcal{O}\Big(\mathrm{e}^{-(n+\frac{1}{2})c|z|}\Big)\sigma_-, & z\in J_i^{o,\frown}\cup J_i^{o,\smile},\\ \mathrm{e}^{-(4(n+\frac{1}{2})\pi\mathrm{i}\int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s)\sigma_3}\Big(\mathrm{I}+\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})c|z-a_j^o|})\sigma_+\Big), & z\in(a_j^o,b_j^o)\setminus\widehat{\mathbb{U}}_{\delta_0^o}(0),\\ \mathrm{e}^{-(4(n+\frac{1}{2})\pi\mathrm{i}\int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s)\sigma_3}\Big(\mathrm{I}+\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})c|z|^{-1}})\sigma_+\Big), & z\in(a_j^o,b_j^o)\cap\widehat{\mathbb{U}}_{\delta_0^o}(0),\\ \mathrm{I}+\mathcal{O}\Big(\mathrm{e}^{-(n+\frac{1}{2})c|z|}\Big)\sigma_+, & z\in\mathfrak{I}\setminus\widehat{\mathbb{U}}_{\delta_0^o}(0),\\ \mathrm{I}+\mathcal{O}\Big(\mathrm{e}^{-(n+\frac{1}{2})c|z|^{-1}}\Big)\sigma_+, & z\in\mathfrak{I}\cap\widehat{\mathbb{U}}_{\delta_0^o}(0), \end{cases}$$

where $c$ (some generic number) $>0$, $\widehat{\mathbb{U}}_{\delta_0^o}(0):=\{z\in\mathbb{C};\ |z|<\delta_0^o\}$, with $\delta_0^o$ some arbitrarily fixed, sufficiently small positive real number, $\mathfrak{I}:=(-\infty,b_0^o)\cup(a_{N+1}^o,+\infty)$, and where the respective convergences are normal, that is, uniform in (respective) compact subsets (see Section 5 below).

Recall from Lemma 2.56 of [1] that, for an oriented skeleton in $\mathbb{C}$ on which the jump matrix of an RHP is defined, one may always choose to add or delete a portion of the skeleton on which the jump matrix equals I without altering the RHP in the operator sense; hence, neglecting those jumps on $\Sigma_o^{\sharp}$ tending exponentially quickly (as $n\to\infty$) to I, and removing the corresponding oriented skeletons from $\Sigma_o^{\sharp}$, it becomes more or less transparent how to construct a parametrix, that is, an approximate solution, of the RHP for $\overset{o}{\mathcal{M}}^{\sharp}\colon\mathbb{C}\setminus\Sigma_o^{\sharp}\to\mathrm{SL}_2(\mathbb{C})$ stated in Lemma 4.2, namely, the large-$n$ solution of the RHP for $\overset{o}{\mathcal{M}}^{\sharp}(z)$ formulated in Lemma 4.2 should be 'close to' the solution of the following limiting, or model, RHP (for $\overset{o}{m}^{\infty}(z)$).

**Lemma 4.3.** *Let the external field* $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *satisfy conditions* (2.3)–(2.5); *furthermore, let* $\widetilde{V}$ *be regular. Let the 'odd' equilibrium measure,* $\mu_V^o$, *and its support,* $\operatorname{supp}(\mu_V^o)=:J_o=\cup_{j=1}^{N+1}J_j^o:=\cup_{j=1}^{N+1}(b_{j-1}^o,a_j^o)$, *be as described in Lemma* 3.5, *and, along with* $\ell_o$ $(\in\mathbb{R})$, *the 'odd' variational constant, satisfy the variational conditions given in Lemma* 3.6, *Equations* (3.9); *moreover, let conditions* (1)–(4) *stated in Lemma* 3.6 *be valid. Then* $\overset{o}{m}^{\infty}\colon\mathbb{C}\setminus J_o^{\infty}\to\mathrm{SL}_2(\mathbb{C})$, *where* $J_o^{\infty}:=J_o\cup(\cup_{j=1}^{N}(a_j^o,b_j^o))$, *solves the following (model)* RHP: (1) $\overset{o}{m}^{\infty}(z)$ *is holomorphic for* $z\in\mathbb{C}\setminus J_o^{\infty}$; (2) $\overset{o}{m}_{\pm}^{\infty}(z):=\lim_{\substack{z'\to z\\ z'\in\pm\text{ side of }J_o^{\infty}}}\overset{o}{m}^{\infty}(z')$ *satisfy the boundary condition*

$$\overset{o}{m}_{+}^{\infty}(z)=\overset{o}{m}_{-}^{\infty}(z)\overset{o}{v}^{\infty}(z),\quad z\in J_o^{\infty},$$

*where*

$$\overset{o}{v}^{\infty}(z)=\begin{cases}\mathrm{i}\sigma_2, & z\in(b_{i-1}^o,a_i^o),\quad i=1,\dots,N+1,\\ \mathrm{e}^{-(4(n+\frac{1}{2})\pi\mathrm{i}\int_{b_j^o}^{a_{N+1}^o}\psi_V^o(s)\,\mathrm{d}s)\sigma_3}, & z\in(a_j^o,b_j^o),\quad j=1,\dots,N;\end{cases}$$

(3) $\overset{o}{m}^{\infty}(z)\underset{\substack{z\to0\\ z\in\mathbb{C}_+\setminus J_o^{\infty}}}{=}(\mathrm{I}+\mathcal{O}(z))\mathbb{E}^{-\sigma_3}$ *and* $\overset{o}{m}^{\infty}(z)\underset{\substack{z\to0\\ z\in\mathbb{C}_-\setminus J_o^{\infty}}}{=}(\mathrm{I}+\mathcal{O}(z))\mathbb{E}^{\sigma_3}$; *and* (4) $\overset{o}{m}^{\infty}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus J_o^{\infty}}}{=}\mathcal{O}(1)$.

The model RHP for $\overset{o}{m}^{\infty}\colon\mathbb{C}\setminus J_o^{\infty}\to\mathrm{SL}_2(\mathbb{C})$ formulated in Lemma 4.3 is (explicitly) solvable in terms of Riemann theta functions (see, for example, Section 3 of [45]; see, also, Section 4.2 of [46]): the solution is succinctly presented below.

**Lemma 4.4.** *Let* $\gamma^o\colon\mathbb{C}\setminus((-\infty,b_0^o)\cup(a_{N+1}^o,+\infty)\cup(\cup_{j=1}^{N}(a_j^o,b_j^o)))\to\mathbb{C}$ *be defined by*

$$\gamma^o(z):=\begin{cases}\left(\left(\dfrac{z-b_0^o}{z-a_{N+1}^o}\right)\displaystyle\prod_{k=1}^{N}\left(\dfrac{z-b_k^o}{z-a_k^o}\right)\right)^{1/4}, & z\in\mathbb{C}_+,\\ -\mathrm{i}\left(\left(\dfrac{z-b_0^o}{z-a_{N+1}^o}\right)\displaystyle\prod_{k=1}^{N}\left(\dfrac{z-b_k^o}{z-a_k^o}\right)\right)^{1/4}, & z\in\mathbb{C}_-,\end{cases}$$

*and set*

$$\gamma^o(0):=\left(\prod_{k=1}^{N+1}b_{k-1}^o(a_k^o)^{-1}\right)^{1/4}\quad(>0).$$

*If* $(\gamma^{o}(0))^{4}=1$*, then, on the lower edge of each finite-length gap, that is,* $(a_{j}^{o},b_{j}^{o})^{-}$*,* $j=1,\dots,N$*,* $(\gamma^{o}(0))^{-1}\gamma^{o}(z)+\gamma^{o}(0)(\gamma^{o}(z))^{-1}=0$ *has exactly one root/zero, and, on the upper edge of each finite-length gap, that is,* $(a_{j}^{o},b_{j}^{o})^{+}$*,* $j=1,\dots,N$*,* $(\gamma^{o}(0))^{-1}\gamma^{o}(z)-\gamma^{o}(0)(\gamma^{o}(z))^{-1}=0$ *has exactly one root/zero; otherwise, if* $(\gamma^{o}(0))^{4}\neq 1$*, there is an additional root/zero in the exterior/unbounded gap* $(-\infty,b_{0}^{o})\cup(a_{N+1}^{o},+\infty)$*. For both cases, label a set of* $N$ *of the lower-edge and upper-edge finite-length-gap roots/zeros as*

$$\left\{z_{j}^{o,\pm}\in(a_{j}^{o},b_{j}^{o})^{\pm}\subset\mathbb{C}_{\pm},\ j=1,\dots,N;\ ((\gamma^{o}(0))^{-1}\gamma^{o}(z)\mp\gamma^{o}(0)(\gamma^{o}(z))^{-1})|_{z=z_{j}^{o,\pm}}=0\right\}$$

*(in the plane,* $z_{j}^{o,+}=z_{j}^{o,-}:=z_{j}^{o}\in(a_{j}^{o},b_{j}^{o})$*,* $j=1,\dots,N$*). Furthermore,* $\gamma^{o}(z)$ *solves the following (scalar)* RHP:

(1) $\gamma^{o}(z)$ *is holomorphic for* $z\in\mathbb{C}\setminus((-\infty,b_{0}^{o})\cup(a_{N+1}^{o},+\infty)\cup(\cup_{j=1}^{N}(a_{j}^{o},b_{j}^{o})))$*;*

(2) $\gamma_{+}^{o}(z)=\gamma_{-}^{o}(z)\mathrm{i}$, $z\in(-\infty,b_{0}^{o})\cup(a_{N+1}^{o},+\infty)\cup(\cup_{j=1}^{N}(a_{j}^{o},b_{j}^{o}))$*;*

(3) $\gamma^{o}(z)\underset{\substack{z\to 0\\ z\in\mathbb{C}_{\pm}}}{=}(-\mathrm{i})^{(1\mp1)/2}\gamma^{o}(0)(1+\mathcal{O}(z))$*; and*

(4) $\gamma^{o}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}_{\pm}}}{=}\mathcal{O}(1)$*.*

*Proof.* Define $\gamma^{o}(z)$ as in the Lemma: then one notes that $(\gamma^{o}(0))^{-1}\gamma^{o}(z)\mp\gamma^{o}(0)(\gamma^{o}(z))^{-1}=0\Leftrightarrow(\gamma^{o}(z))^{2}\mp(\gamma^{o}(0))^{2}=0\Rightarrow(\gamma^{o}(z))^{4}-(\gamma^{o}(0))^{4}=0\Leftrightarrow\mathcal{Q}^{o}(z)\ (\in\mathbb{R}[z]):=(z-b_{0}^{o})\prod_{k=1}^{N}(z-b_{k}^{o})-(\gamma^{o}(0))^{4}(z-a_{N+1}^{o})\prod_{k=1}^{N}(z-a_{k}^{o})=0$, whence, via a straightforward calculation, and using the fact that $(\gamma^{o}(0))^{4}=\prod_{k=1}^{N+1}b_{k-1}^{o}(a_{k}^{o})^{-1}>0$, one shows that $\mathcal{Q}^{o}(a_{j}^{o})=(-1)^{N-j+1}\widehat{\mathcal{Q}}_{a_{j}^{o}}^{o}$, $j=1,\dots,N$, where $\widehat{\mathcal{Q}}_{a_{j}^{o}}^{o}:=(b_{j}^{o}-a_{j}^{o})(a_{j}^{o}-b_{0}^{o})\prod_{k=1}^{j-1}(a_{j}^{o}-b_{k}^{o})\prod_{l=j+1}^{N}(b_{l}^{o}-a_{j}^{o})>0$, and $\mathcal{Q}^{o}(b_{j}^{o})=-(-1)^{N-j+1}\widehat{\mathcal{Q}}_{b_{j}^{o}}^{o}$, $j=1,\dots,N$, where $\widehat{\mathcal{Q}}_{b_{j}^{o}}^{o}:=(\gamma^{o}(0))^{4}(b_{j}^{o}-a_{j}^{o})(a_{N+1}^{o}-b_{j}^{o})\prod_{k=1}^{j-1}(b_{j}^{o}-a_{k}^{o})\prod_{l=j+1}^{N}(a_{l}^{o}-b_{j}^{o})>0$; thus, $\mathcal{Q}^{o}(a_{j}^{o})\mathcal{Q}^{o}(b_{j}^{o})<0$, $j=1,\dots,N$, which shows that: (i) for $(\gamma^{o}(0))^{4}\neq 1$, since $\deg(\mathcal{Q}^{o}(z))=N+1$, there are $N+1$ (simple) roots/zeros of $\mathcal{Q}^{o}(z)$, one in each (open) finite-length gap $(a_{j}^{o},b_{j}^{o})$, $j=1,\dots,N$, and one in the (open) unbounded/exterior gap $(-\infty,b_{0}^{o})\cup(a_{N+1}^{o},+\infty)$; and (ii) for $(\gamma^{o}(0))^{4}=1$, since $\deg(\mathcal{Q}^{o}(z))=N$, there are $N$ (simple) roots/zeros of $\mathcal{Q}^{o}(z)$, one in each (open) finite-length gap $(a_{j}^{o},b_{j}^{o})$, $j=1,\dots,N$. For both cases, label a set of $N$ of the roots/zeros of $\mathcal{Q}^{o}(z)$ as $\{z_{j}^{o}\}_{j=1}^{N}$. A straightforward analysis of the branch cuts shows that, for $z\in\cup_{j=1}^{N}(a_{j}^{o},b_{j}^{o})^{\pm}$, $\pm(\gamma^{o}(z))^{2}>0$, whence $\{z_{j}^{o,\pm}\}_{j=1}^{N}=\{z^{\pm}\in(a_{j}^{o},b_{j}^{o})^{\pm}\subset\mathbb{C}_{\pm},\ j=1,\dots,N;\ ((\gamma^{o}(0))^{-1}\gamma^{o}(z)\mp\gamma^{o}(0)(\gamma^{o}(z))^{-1})|_{z=z^{\pm}}=0\}$. Setting $\widetilde{J^{o}}:=(-\infty,b_{0}^{o})\cup(a_{N+1}^{o},+\infty)\cup(\cup_{j=1}^{N}(a_{j}^{o},b_{j}^{o}))$, one shows, upon performing a straightforward analysis of the branch cuts, that $\gamma^{o}(z)$ solves the RHP $(\gamma^{o}(z),\mathrm{i},\widetilde{J^{o}})$ formulated in the Lemma. ☐

All of the notation/nomenclature used in Lemma 4.5 below has been defined at the end of Subsection 2.1; the reader, therefore, is advised to peruse the relevant notations(s), etc., before proceeding to Lemma 4.5. Let $\mathcal{Y}_{o}$ denote the Riemann surface of $y^{2}=R_{o}(z)=\prod_{k=1}^{N+1}(z-b_{k-1}^{o})(z-a_{k}^{o})$, where the single-valued branch of the square root is chosen so that $z^{-(N+1)}(R_{o}(z))^{1/2}\underset{\substack{z\to\infty\\ z\in\mathbb{C}_{\pm}}}{\sim}\pm1$. Let $\mathcal{P}:=(y,z)$ denote a point on the Riemann surface $\mathcal{Y}_{o}$ $(:=\{(y,z);\ y^{2}=R_{o}(z)\})$. The notation $0^{\pm}$ (used in Lemma 4.5 below) means: $\mathcal{P}\to 0^{\pm}\Leftrightarrow z\to 0,\ y\sim\pm(-1)^{\mathcal{N}_{+}}(\prod_{k=1}^{N+1}|b_{k-1}^{o}a_{k}^{o}|)^{1/2}$, where $\mathcal{N}_{+}\in\{0,\dots,N+1\}$ is the number of bands to the right of $z=0$.

**Lemma 4.5.** *Let* $\overset{o}{m}{}^{\infty}:\mathbb{C}\setminus J_{o}^{\infty}\to\mathrm{SL}_{2}(\mathbb{C})$ *solve the* RHP *formulated in Lemma* 4.3*. Then,*

$$\overset{o}{m}{}^{\infty}(z)=\begin{cases}\overset{o}{\mathfrak{M}}{}^{\infty}(z), & z\in\mathbb{C}_{+},\\ -\mathrm{i}\,\overset{o}{\mathfrak{M}}{}^{\infty}(z)\sigma_{2}, & z\in\mathbb{C}_{-},\end{cases}$$

*where*

$$\overset{o}{\mathfrak{M}}{}^{\infty}(z):=\mathbb{E}^{-\sigma_{3}}\begin{pmatrix}\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})} & 0\\ 0 & \frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\end{pmatrix}\overset{o}{\boldsymbol{\Theta}}{}_{\natural}^{\infty}(z),$$

*and*

$$\overset{o}{\boldsymbol{\Theta}}{}_{\natural}^{\infty}(z)=\begin{pmatrix}\frac{((\gamma^{o}(0))^{-1}\gamma^{o}(z)+\gamma^{o}(0)(\gamma^{o}(z))^{-1})}{2}\overset{o}{\boldsymbol{\Theta}}{}_{11}^{\infty}(z) & -\frac{((\gamma^{o}(0))^{-1}\gamma^{o}(z)-\gamma^{o}(0)(\gamma^{o}(z))^{-1})}{2\mathrm{i}}\overset{o}{\boldsymbol{\Theta}}{}_{12}^{\infty}(z)\\ \frac{((\gamma^{o}(0))^{-1}\gamma^{o}(z)-\gamma^{o}(0)(\gamma^{o}(z))^{-1})}{2\mathrm{i}}\overset{o}{\boldsymbol{\Theta}}{}_{21}^{\infty}(z) & \frac{((\gamma^{o}(0))^{-1}\gamma^{o}(z)+\gamma^{o}(0)(\gamma^{o}(z))^{-1})}{2}\overset{o}{\boldsymbol{\Theta}}{}_{22}^{\infty}(z)\end{pmatrix},$$

$$\overset{o}{\boldsymbol{\Theta}}{}^{\infty}_{11}(z):=\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)+\boldsymbol{d}_{o})}, \qquad \overset{o}{\boldsymbol{\Theta}}{}^{\infty}_{12}(z):=\frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}(z)+\boldsymbol{d}_{o})},$$

$$\overset{o}{\boldsymbol{\Theta}}{}^{\infty}_{21}(z):=\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)-\boldsymbol{d}_{o})}, \qquad \overset{o}{\boldsymbol{\Theta}}{}^{\infty}_{22}(z):=\frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)+\boldsymbol{d}_{o})},$$

*with* $\gamma^{o}(z)$ *characterised completely in Lemma* 4.4, $\boldsymbol{\Omega}^{o}:=(\Omega^{o}_{1},\Omega^{o}_{2},\dots,\Omega^{o}_{N})^{\mathrm{T}}$ $(\in\mathbb{R}^{N})$, *where* $\Omega^{o}_{j}=4\pi\int_{b^{o}_{j}}^{a^{o}_{N+1}}\psi^{o}_{V}(s)\,\mathrm{d}s$, $j=1,\dots,N$, *and* $^{\mathrm{T}}$ *denotes transposition*, $\boldsymbol{d}_{o}\equiv-\sum_{j=1}^{N}\int_{a^{o}_{j}}^{z^{o,-}_{j}}\boldsymbol{\omega}^{o}$ $(=\sum_{j=1}^{N}\int_{a^{o}_{j}}^{z^{o,+}_{j}}\boldsymbol{\omega}^{o})$, $\{z^{o,\pm}_{j}\}_{j=1}^{N}$ *are characterised completely in Lemma* 4.4, $\boldsymbol{\omega}^{o}$ *is the associated normalised basis of holomorphic one-forms of* $\mathcal{Y}_{o}$, $\boldsymbol{u}^{o}(z):=\int_{a^{o}_{N+1}}^{z}\boldsymbol{\omega}^{o}$ $(\in\mathrm{Jac}(\mathcal{Y}_{o}))$, *and* $\boldsymbol{u}^{o}_{+}(0):=\int_{a^{o}_{N+1}}^{0^{+}}\boldsymbol{\omega}^{o}$; *furthermore, the solution is unique.*

*Proof.* Let $\overset{o}{m}{}^{\infty}\colon\mathbb{C}\setminus J^{\infty}_{o}\to\mathrm{SL}_{2}(\mathbb{C})$ solve the RHP formulated in Lemma 4.3, and define $\overset{o}{m}{}^{\infty}(z)$, in terms of $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$, as in the Lemma. A straightforward calculation shows that $\overset{o}{\mathfrak{M}}{}^{\infty}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_{2}(\mathbb{C})$ solves the following 'twisted' RHP: (i) $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$ is holomorphic for $z\in\mathbb{C}\setminus\widetilde{J}^{o}$, where $\widetilde{J}^{o}:=(-\infty,b^{o}_{0})\cup(a^{o}_{N+1},+\infty)\cup(\cup_{j=1}^{N}(a^{o}_{j},b^{o}_{j}))$; (ii) $\overset{o}{\mathfrak{M}}{}^{\infty}_{\pm}(z):=\lim_{\substack{z'\to z\\ z'\in\pm\text{ side of }\widetilde{J}^{o}}}\overset{o}{\mathfrak{M}}{}^{\infty}(z')$ satisfy the boundary condition $\overset{o}{\mathfrak{M}}{}^{\infty}_{+}(z)=\overset{o}{\mathfrak{M}}{}^{\infty}_{-}(z)\overset{o}{\mathcal{V}}{}^{\infty}(z)$, $z\in\widetilde{J}^{o}$, where

$$\overset{o}{\mathcal{V}}{}^{\infty}(z):=\begin{cases}\mathrm{I}, & z\in J_{o},\\ -\mathrm{i}\sigma_{2}, & z\in(-\infty,b^{o}_{0})\cup(a^{o}_{N+1},+\infty),\\ -\mathrm{i}\sigma_{2}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega^{o}_{j}\sigma_{3}}, & z\in(a^{o}_{j},b^{o}_{j}),\quad j=1,\dots,N,\end{cases} \tag{4.1}$$

with $\Omega^{o}_{j}=4\pi\int_{b^{o}_{j}}^{a^{o}_{N+1}}\psi^{o}_{V}(s)\,\mathrm{d}s$, $j=1,\dots,N$; (iii) $\overset{o}{\mathfrak{M}}{}^{\infty}(z)\underset{\substack{z\to0\\ z\in\mathbb{C}_{+}}}{=}\mathbb{E}^{-\sigma_{3}}+\mathcal{O}(z)$ and $\overset{o}{\mathfrak{M}}{}^{\infty}(z)\underset{\substack{z\to0\\ z\in\mathbb{C}_{-}}}{=}\mathrm{i}\mathbb{E}^{\sigma_{3}}\sigma_{2}+\mathcal{O}(z)$; and (iv) $\overset{o}{\mathfrak{M}}{}^{\infty}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\widetilde{J}^{o}}}{=}\mathcal{O}(1)$. The solution of this latter (twisted) RHP for $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$ is constructed out of the solution of two, simpler RHPs: $(\mathcal{N}^{o}(z),-\mathrm{i}\sigma_{2},\widetilde{J}^{o})$ and $(\overset{o}{\mathfrak{m}}{}^{\infty}(z),\overset{o}{\mathcal{U}}{}^{\infty}(z),\widetilde{J}^{o})$, where $\overset{o}{\mathcal{U}}{}^{\infty}(z)$ equals $\exp(\mathrm{i}(n+1/2)\Omega^{o}_{j}\sigma_{3})\sigma_{1}$ for $z\in(a^{o}_{j},b^{o}_{j})$, $j=1,\dots,N$, and equals $\mathrm{I}$ for $z\in(-\infty,b^{o}_{0})\cup(a^{o}_{N+1},+\infty)$. The RHP $(\mathcal{N}^{o}(z),-\mathrm{i}\sigma_{2},\widetilde{J}^{o})$ is solved explicitly by diagonalising the jump matrix, and thus reduced to two scalar RHPs [2] (see, also, [45,47,79]): the solution is

$$\mathcal{N}^{o}(z)=\begin{pmatrix}\frac{1}{2}((\gamma^{o}(0))^{-1}\gamma^{o}(z)+\gamma^{o}(0)(\gamma^{o}(z))^{-1}) & -\frac{1}{2\mathrm{i}}((\gamma^{o}(0))^{-1}\gamma^{o}(z)-\gamma^{o}(0)(\gamma^{o}(z))^{-1})\\ \frac{1}{2\mathrm{i}}((\gamma^{o}(0))^{-1}\gamma^{o}(z)-\gamma^{o}(0)(\gamma^{o}(z))^{-1}) & \frac{1}{2}((\gamma^{o}(0))^{-1}\gamma^{o}(z)+\gamma^{o}(0)(\gamma^{o}(z))^{-1})\end{pmatrix},$$

where $\gamma^{o}\colon\mathbb{C}\setminus\widetilde{J}^{o}\to\mathbb{C}$ is characterised completely in Lemma 4.4; furthermore, $\mathcal{N}^{o}(z)$ is piecewise (sectionally) holomorphic for $z\in\mathbb{C}\setminus\widetilde{J}^{o}$, and $\mathcal{N}^{o}(z)\underset{\substack{z\to0\\ z\in\mathbb{C}_{+}}}{=}\mathrm{I}+\mathcal{O}(z)$ and $\mathcal{N}^{o}(z)\underset{\substack{z\to0\\ z\in\mathbb{C}_{-}}}{=}\mathrm{i}\sigma_{2}+\mathcal{O}(z)$[13].

Consider, now, the functions $\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)\pm\boldsymbol{d}_{o})$, where $\boldsymbol{u}^{o}(z)\colon z\to\mathrm{Jac}(\mathcal{Y}_{o})$, $z\mapsto\boldsymbol{u}^{o}(z):=\int_{a^{o}_{N+1}}^{z}\boldsymbol{\omega}^{o}$, with $\boldsymbol{\omega}^{o}$ the associated normalised basis of holomorphic one-forms of $\mathcal{Y}_{o}$, $\boldsymbol{d}_{o}\equiv-\sum_{j=1}^{N}\int_{a^{o}_{j}}^{z^{o,-}_{j}}\boldsymbol{\omega}^{o}=\sum_{j=1}^{N}\int_{a^{o}_{j}}^{z^{o,+}_{j}}\boldsymbol{\omega}^{o}$, where $\equiv$ denotes equivalence modulo the period lattice, and $\{z^{o,\pm}_{j}\}_{j=1}^{N}$ are characterised completely in Lemma 4.4. From the general theory of theta functions on Riemann surfaces (see, for example, [77,78]), $\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)+\boldsymbol{d}_{o})$, for $z\in\mathcal{Y}_{o}:=\{(y,z);\ y^{2}=\prod_{k=1}^{N+1}(z-b^{o}_{k-1})(z-a^{o}_{k})\}$, is either identically zero on $\mathcal{Y}_{o}$ or has precisely $N$ (simple) zeros (the generic case). In this case, since the divisors $\prod_{j=1}^{N}z^{o,-}_{j}$ and $\prod_{j=1}^{N}z^{o,+}_{j}$ are non-special, one uses Lemma 3.27 of [45] (see, also, Lemma 4.2 of [46]) and the representation [78] $\boldsymbol{K}_{o}=\sum_{j=1}^{N}\int_{a^{o}_{j}}^{a^{o}_{N+1}}\boldsymbol{\omega}^{o}$, for the 'odd' vector of Riemann constants, with $2\boldsymbol{K}_{o}=0$ and $s\boldsymbol{K}_{o}\neq0$, $0<s<2$, to

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[13]Note that, strictly speaking, $\mathcal{N}^{o}(z)$, as given above, does not solve the RHP $(\mathcal{N}^{o}(z),-\mathrm{i}\sigma_{2},\widetilde{J}^{o})$ in the sense defined heretofore, as $\mathcal{N}^{o}\restriction_{\mathbb{C}_{\pm}}$ can not be extended continuously to $\overline{\mathbb{C}}_{\pm}$; however, $\mathcal{N}^{o}(\cdot\pm\mathrm{i}\varepsilon)$ converge in $\mathcal{L}^{2}_{\mathrm{M}_{2}(\mathbb{C}),\mathrm{loc}}(\mathbb{R})$ as $\varepsilon\downarrow0$ to $\mathrm{SL}_{2}(\mathbb{C})$-valued functions $\mathcal{N}^{o}(z)$ in $\mathcal{L}^{2}_{\mathrm{M}_{2}(\mathbb{C})}(\widetilde{J}^{o})$ that satisfy $\mathcal{N}^{o}_{+}(z)=\mathcal{N}^{o}_{-}(z)(-\mathrm{i}\sigma_{2})$ a.e. on $\widetilde{J}^{o}$: one then shows that $\mathcal{N}^{o}(z)$ is the unique solution of the RHP $(\mathcal{N}^{o}(z),-\mathrm{i}\sigma_{2},\widetilde{J}^{o})$, where the latter boundary/jump condition is interpreted in the $\mathcal{L}^{2}_{\mathrm{M}_{2}(\mathbb{C}),\mathrm{loc}}$ sense.

arrive at

$$\begin{aligned}\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)+\boldsymbol{d}_{o}) &= \boldsymbol{\theta}^{o}\left(\boldsymbol{u}^{o}(z)-\sum_{j=1}^{N}\int_{a_{j}^{o}}^{z_{j}^{o,-}}\boldsymbol{\omega}^{o}\right)=\boldsymbol{\theta}^{o}\left(\int_{a_{N+1}^{o}}^{z}\boldsymbol{\omega}^{o}-\boldsymbol{K}_{o}-\sum_{j=1}^{N}\int_{a_{N+1}^{o}}^{z_{j}^{o,-}}\boldsymbol{\omega}^{o}\right)=0\\ &\Leftrightarrow z\in\left\{z_{1}^{o,-},z_{2}^{o,-},\dots,z_{N}^{o,-}\right\},\\ \boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)-\boldsymbol{d}_{o}) &= \boldsymbol{\theta}^{o}\left(\boldsymbol{u}^{o}(z)-\sum_{j=1}^{N}\int_{a_{j}^{o}}^{z_{j}^{o,+}}\boldsymbol{\omega}^{o}\right)=\boldsymbol{\theta}^{o}\left(\int_{a_{N+1}^{o}}^{z}\boldsymbol{\omega}^{o}-\boldsymbol{K}_{o}-\sum_{j=1}^{N}\int_{a_{N+1}^{o}}^{z_{j}^{o,+}}\boldsymbol{\omega}^{o}\right)=0\\ &\Leftrightarrow z\in\left\{z_{1}^{o,+},z_{2}^{o,+},\dots,z_{N}^{o,+}\right\}.\end{aligned}$$

Following Lemma 3.21 of [45], set

$$\overset{o}{\mathfrak{m}}^{\infty}(z):=\begin{pmatrix}\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)+\boldsymbol{d}_{o})} & \frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}(z)+\boldsymbol{d}_{o})}\\ \frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)-\boldsymbol{d}_{o})} & \frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z)+\boldsymbol{d}_{o})}\end{pmatrix},$$

where $\boldsymbol{\Omega}^{o}:=(\Omega_{1}^{o},\Omega_{2}^{o},\dots,\Omega_{N}^{o})^{\mathrm{T}}$ $(\in\mathbb{R}^{N})$, with $\Omega_{j}^{o}$, $j=1,\dots,N$, given above, and $^{\mathrm{T}}$ denoting transposition. Using Lemma 3.18 of [45] (or, equivalently, Equations (4.65) and (4.66) of [46]), that is, for $z\in(a_{j}^{o},b_{j}^{o})$, $j=1,\dots,N$, $\boldsymbol{u}_{+}^{o}(z)+\boldsymbol{u}_{-}^{o}(z)\equiv-\tau_{j}^{o}$ $(:=-\tau^{o}e_{j})$, $j=1,\dots,N$, with $\tau^{o}:=(\tau^{o})_{i,j=1,\dots,N}:=(\oint_{\boldsymbol{\beta}_{j}^{o}}\omega_{i}^{o})_{i,j=1,\dots,N}$ (the associated matrix of Riemann periods), and, for $z\in(-\infty,b_{0}^{o})\cup(a_{N+1}^{o},+\infty)$, $\boldsymbol{u}_{+}^{o}(z)+\boldsymbol{u}_{-}^{o}(z)\equiv 0$, where $\boldsymbol{u}_{\pm}^{o}(z):=\int_{a_{N+1}^{o}}^{z^{\pm}}\boldsymbol{\omega}^{o}$, with $z^{\pm}\in(a_{j}^{o},b_{j}^{o})^{\pm}$, $j=1,\dots,N$, and the evenness and (quasi-) periodicity properties of $\boldsymbol{\theta}^{o}$, one shows that, for $z\in(a_{j}^{o},b_{j}^{o})$, $j=1,\dots,N$,

$$\begin{aligned}\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(z)+\boldsymbol{d}_{o})}&=\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_{j}^{o}}\frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{-}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{-}^{o}(z)+\boldsymbol{d}_{o})},\\ \frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(z)-\boldsymbol{d}_{o})}&=\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_{j}^{o}}\frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{-}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{-}^{o}(z)+\boldsymbol{d}_{o})},\\ \frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{+}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{+}^{o}(z)+\boldsymbol{d}_{o})}&=\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_{j}^{o}}\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{-}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{-}^{o}(z)+\boldsymbol{d}_{o})},\\ \frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{+}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(z)+\boldsymbol{d}_{o})}&=\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_{j}^{o}}\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{-}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{-}^{o}(z)-\boldsymbol{d}_{o})},\end{aligned}$$

and, for $z\in(-\infty,b_{0}^{o})\cup(a_{N+1}^{o},+\infty)$, one obtains the same relations as above but with the changes $\exp(\mp\mathrm{i}(n+1/2)\Omega_{j}^{o})\to 1$. Set, as in Proposition 3.31 of [45],

$$\overset{o}{\mathcal{Q}}^{\infty}(z):=\begin{pmatrix}(\mathcal{N}^{o}(z))_{11}(\overset{o}{\mathfrak{m}}^{\infty}(z))_{11} & (\mathcal{N}^{o}(z))_{12}(\overset{o}{\mathfrak{m}}^{\infty}(z))_{12}\\ (\mathcal{N}^{o}(z))_{21}(\overset{o}{\mathfrak{m}}^{\infty}(z))_{21} & (\mathcal{N}^{o}(z))_{22}(\overset{o}{\mathfrak{m}}^{\infty}(z))_{22}\end{pmatrix},$$

where $(*)_{ij}$, $i,j=1,2$, denotes the $(i\,j)$-element of $(*)$. Recalling that $\mathcal{N}^{o}\colon\mathbb{C}\setminus\widetilde{J}^{o}\to\mathrm{SL}_{2}(\mathbb{C})$ solves the RHP $(\mathcal{N}^{o}(z),-\mathrm{i}\sigma_{2},\widetilde{J}^{o})$, using the above theta-functional relations and the small-$z$ asymptotic expansion of $\boldsymbol{u}^{o}(z)$ (see Section 5, the proof of Proposition 5.3), one shows that $\overset{o}{\mathcal{Q}}^{\infty}(z)$ solves the following RHP: (i) $\overset{o}{\mathcal{Q}}^{\infty}(z)$ is holomorphic for $z\in\mathbb{C}\setminus\widetilde{J}^{o}$; (ii) $\overset{o}{\mathcal{Q}}_{\pm}^{\infty}(z):=\lim_{\substack{z'\to z\\ z'\,\in\,\pm\,\text{side of}\,\widetilde{J}^{o}}}\overset{o}{\mathcal{Q}}^{\infty}(z')$ satisfy the boundary condition $\overset{o}{\mathcal{Q}}_{+}^{\infty}(z)=\overset{o}{\mathcal{Q}}_{-}^{\infty}(z)\overset{o}{\mathcal{V}}^{\infty}(z)$, $z\in\widetilde{J}^{o}$, where $\overset{o}{\mathcal{V}}^{\infty}(z)$ is defined in Equation (4.1); (iii)

$$\begin{aligned}\overset{o}{\mathcal{Q}}^{\infty}(z)\underset{\substack{z\to 0\\ z\in\mathbb{C}_{+}}}{=}&\begin{pmatrix}\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})} & 0\\ 0 & \frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})}\end{pmatrix}+\mathcal{O}(z),\\ \overset{o}{\mathcal{Q}}^{\infty}(z)\underset{\substack{z\to 0\\ z\in\mathbb{C}_{-}}}{=}&\begin{pmatrix}0 & \frac{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{-}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{-}^{o}(0)+\boldsymbol{d}_{o})}\\ -\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{-}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{-}^{o}(0)-\boldsymbol{d}_{o})} & 0\end{pmatrix}+\mathcal{O}(z),\end{aligned}$$

where $\boldsymbol{u}^{o}_{\pm}(0):=\int_{a^{o}_{N+1}}^{0^{\pm}}\boldsymbol{\omega}^{o}$; and (iv) $\overset{o}{\mathcal{Q}}{}^{\infty}(z)\underset{\substack{z\to\infty\\ z\in\mathbb{C}\setminus\widetilde{J}^{o}}}{=}\mathcal{O}(1)$. Now, using the fact that, for $z\in(a^{o}_{j},b^{o}_{j})$, $j=1,\dots,N$, $\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{u}^{o}_{-}(0)=\int_{a^{o}_{N+1}}^{0^{+}}\boldsymbol{\omega}^{o}+\int_{a^{o}_{N+1}}^{0^{-}}\boldsymbol{\omega}^{o}\equiv-\tau^{o}_{j}$, $j=1,\dots,N$, and, for $z\in(-\infty,b^{o}_{0})\cup(a^{o}_{N+1},+\infty)$, $\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{u}^{o}_{-}(0)=\int_{a^{o}_{N+1}}^{0^{+}}\boldsymbol{\omega}^{o}+\int_{a^{o}_{N+1}}^{0^{-}}\boldsymbol{\omega}^{o}\equiv0$, upon multiplying $\overset{o}{\mathcal{Q}}{}^{\infty}(z)$ on the left by

$$\operatorname{diag}\left(\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})},\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\right)=:\overset{o}{\mathfrak{c}}{}^{\infty},$$

that is, $\overset{o}{\mathcal{Q}}{}^{\infty}(z)\to\overset{o}{\mathfrak{c}}{}^{\infty}\overset{o}{\mathcal{Q}}{}^{\infty}(z)=:\mathcal{M}^{\infty}_{o}(z)$, one shows that $\mathcal{M}^{\infty}_{o}\colon\mathbb{C}\setminus\widetilde{J}^{o}\to\mathrm{SL}_{2}(\mathbb{C})$ solves the RHP $(\mathcal{M}^{\infty}_{o}(z),\overset{o}{\mathcal{V}}{}^{\infty}(z),\widetilde{J}^{o})$. Using, finally, the formula $\overset{o}{m}{}^{\infty}(z)=\begin{cases}\overset{o}{\mathfrak{M}}{}^{\infty}(z), & z\in\mathbb{C}_{+},\\ -\mathrm{i}\,\overset{o}{\mathfrak{M}}{}^{\infty}(z)\sigma_{2}, & z\in\mathbb{C}_{-},\end{cases}$ one shows that $\overset{o}{m}{}^{\infty}\colon\mathbb{C}\setminus J^{\infty}_{o}\to\mathrm{SL}_{2}(\mathbb{C})$ solves the model RHP formulated in Lemma 4.3. One notes from the formula for $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$ stated in the Lemma that it is well defined for $\mathbb{C}\setminus\mathbb{R}$; in particular, it is single valued and analytic (see below) for $z\in\mathbb{C}\setminus\widetilde{J}^{o}$ (independently of the path in $\mathbb{C}\setminus\widetilde{J}^{o}$ chosen to evaluate $\boldsymbol{u}^{o}(z)=\int_{a^{o}_{N+1}}^{z}\boldsymbol{\omega}^{o}$). Furthermore (cf. Lemma 4.4 and the analysis above), since $\{z'\in\mathbb{C};\,\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z')\pm\boldsymbol{d}_{o})=0\}=\{z^{o,\mp}_{j}\}_{j=1}^{N}=\{z'\in\mathbb{C};\,((\gamma^{o}(0))^{-1}\gamma^{o}(z)\pm\gamma^{o}(0)(\gamma^{o}(z))^{-1})|_{z=z'}=0\}$, one notes that the (simple) poles of $(\overset{o}{\mathfrak{m}}{}^{\infty}(z))_{11}$ and $(\overset{o}{\mathfrak{m}}{}^{\infty}(z))_{22}$ (resp., $(\overset{o}{\mathfrak{m}}{}^{\infty}(z))_{12}$ and $(\overset{o}{\mathfrak{m}}{}^{\infty}(z))_{21}$), that is, $\{z'\in\mathbb{C};\,\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z')+\boldsymbol{d}_{o})=0\}$ (resp., $\{z'\in\mathbb{C};\,\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}(z')-\boldsymbol{d}_{o})=0\}$), are exactly cancelled by the (simple) zeros of $(\gamma^{o}(0))^{-1}\gamma^{o}(z)+\gamma^{o}(0)(\gamma^{o}(z))^{-1}$ (resp., $(\gamma^{o}(0))^{-1}\gamma^{o}(z)-\gamma^{o}(0)(\gamma^{o}(z))^{-1}$); thus, $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$ has only $\frac{1}{4}$-root singularities at the end-points of the support of the 'odd' equilibrium measure, $\{b^{o}_{j-1},a^{o}_{j}\}_{j=1}^{N+1}$. (This shows that $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$ obtains its boundary values, $\overset{o}{\mathfrak{M}}{}^{\infty}_{\pm}(z):=\lim_{\varepsilon\downarrow0}\overset{o}{\mathfrak{M}}{}^{\infty}(z\pm\mathrm{i}\varepsilon)$, in the $\mathcal{L}^{2}_{\mathrm{M}_{2}(\mathbb{C})}(\mathbb{R})$ sense.) From the definition of $\overset{o}{m}{}^{\infty}(z)$ in terms of $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$ given in the Lemma, the explicit formula for $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$, and recalling that $\overset{o}{m}{}^{\infty}(z)$ solves the model RHP formulated in Lemma 4.3, one learns that, as $\det(\overset{o}{\upsilon}{}^{\infty}(z))=1$, $\det(\overset{o}{m}{}^{\infty}_{+}(z))=\det(\overset{o}{m}{}^{\infty}_{-}(z))$, that is, $\det(\overset{o}{m}{}^{\infty}(z))$ has no 'jumps', whence $\det(\overset{o}{m}{}^{\infty}(z))$ has, at worst, (isolated) $\frac{1}{2}$-root singularities at $\{b^{o}_{j-1},a^{o}_{j}\}_{j=1}^{N+1}$, which are removable, which implies that $\det(\overset{o}{m}{}^{\infty}(z))$ is entire and bounded; hence, via a generalisation of Liouville's Theorem, and the asymptotic relation $\det(\overset{o}{m}{}^{\infty}(z))\underset{\substack{z\to0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=}1+\mathcal{O}(z)$, one arrives at $\det(\overset{o}{m}{}^{\infty}(z))=1\Rightarrow\overset{o}{m}{}^{\infty}\in\mathrm{SL}_{2}(\mathbb{C})$. Also, from the definition of $\overset{o}{m}{}^{\infty}(z)$ in terms of $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$ and the explicit formula for $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$, it follows that both $\overset{o}{m}{}^{\infty}(z)$ and $(\overset{o}{m}{}^{\infty}(z))^{-1}$ are uniformly bounded as functions of $n$ (as $n\to\infty$) for $z$ in compact subsets away from $\{b^{o}_{j-1},a^{o}_{j}\}_{j=1}^{N+1}$.

Let $\mathcal{S}^{\infty}_{o}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_{2}(\mathbb{C})$ be another solution of the RHP $(\overset{o}{\mathfrak{M}}{}^{\infty}(z),\overset{o}{\mathcal{V}}{}^{\infty}(z),\mathbb{R})$ formulated at the beginning of the proof. Set $\Delta^{o}(z):=\mathcal{S}^{\infty}_{o}(z)(\overset{o}{\mathfrak{M}}{}^{\infty}(z))^{-1}$. Then $\Delta^{o}_{+}(z)=(\mathcal{S}^{\infty}_{o}(z))_{+}(\overset{o}{\mathfrak{M}}{}^{\infty}_{+}(z))^{-1}=(\mathcal{S}^{\infty}_{o}(z))_{-}\cdot\overset{o}{\mathcal{V}}{}^{\infty}(z)(\overset{o}{\mathfrak{M}}{}^{\infty}_{-}(z)\overset{o}{\mathcal{V}}{}^{\infty}(z))^{-1}=(\mathcal{S}^{\infty}_{o}(z))_{-}(\overset{o}{\mathfrak{M}}{}^{\infty}_{-}(z))^{-1}=\Delta^{o}_{-}(z)$, hence $\Delta^{o}(z)$ is analytic across $\mathbb{R}$; moreover, as $\det(\overset{o}{\mathfrak{M}}{}^{\infty}(z))=1$, it follows that $\Delta^{o}(z)$ has, at worst, $\mathcal{L}^{1}_{\mathrm{M}_{2}(\mathbb{C})}(*)$-singularities at $b^{o}_{j-1},a^{o}_{j}$, $j=1,\dots,N+1$, which, as per the discussion above, are removable; thence, noting that $\Delta^{o}(z)\to\mathrm{I}$ as $z\to0$ $(z\in\mathbb{C}\setminus\mathbb{R})$, one concludes that $\Delta^{o}(z)=\mathrm{I}$, whence $\mathcal{S}^{\infty}_{o}(z)=\overset{o}{\mathfrak{M}}{}^{\infty}(z)$. □

In order to prove that there is a solution of the (full) RHP $(\overset{o}{\mathcal{M}}{}^{\sharp}(z),\overset{o}{\upsilon}{}^{\sharp}(z),\Sigma^{\sharp}_{o})$, formulated in Lemma 4.2, close to the parametrix, one needs to know that the parametrix is *uniformly* bounded: more precisely, by (certain) general theorems (see, for example, [86]), one needs to know that $\overset{o}{\upsilon}{}^{\sharp}(z)\to\overset{o}{\upsilon}{}^{\infty}(z)$ as $n\to\infty$ uniformly for $z\in\Sigma^{\sharp}_{o}$ in the $\mathcal{L}^{2}_{\mathrm{M}_{2}(\mathbb{C})}(\Sigma^{\sharp}_{o})\cap\mathcal{L}^{\infty}_{\mathrm{M}_{2}(\mathbb{C})}(\Sigma^{\sharp}_{o})$ sense, that is, uniformly,

$$\lim_{n\to\infty}\|\overset{o}{\upsilon}{}^{\sharp}(\cdot)-\overset{o}{\upsilon}{}^{\infty}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_{2}(\mathbb{C})}(\Sigma^{\sharp}_{o})\cap\mathcal{L}^{\infty}_{\mathrm{M}_{2}(\mathbb{C})}(\Sigma^{\sharp}_{o})}:=\lim_{n\to\infty}\sum_{p\in\{2,\infty\}}\|\overset{o}{\upsilon}{}^{\sharp}(\cdot)-\overset{o}{\upsilon}{}^{\infty}(\cdot)\|_{\mathcal{L}^{p}_{\mathrm{M}_{2}(\mathbb{C})}(\Sigma^{\sharp}_{o})}=0;$$

however, notwithstanding the fact that $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is regular ($h^{o}_{V}(b^{o}_{j-1}),h^{o}_{V}(a^{o}_{j})\neq0$, $j=1,\dots,N+1$), since the strict inequalities $g^{o}_{+}(z)+g^{o}_{-}(z)-\widetilde{V}(z)-\ell_{o}-\mathfrak{Q}^{+}_{\mathscr{A}}-\mathfrak{Q}^{-}_{\mathscr{A}}<0$, $z\in(-\infty,b^{o}_{0})\cup(a^{o}_{N+1},+\infty)\cup(\cup_{j=1}^{N}(a^{o}_{j},b^{o}_{j}))$,

and $\pm\mathrm{Re}(\mathrm{i}\int_{z}^{a_{N+1}^{o}}\psi_{V}^{o}(s)\,\mathrm{d}s) > 0$, $z \in \mathbb{C}_{\pm} \cap (\cup_{j=1}^{N+1}\mathbb{U}_{j}^{o})$, fail at the end-points of the support of the 'odd' equilibrium measure, this implies that $\overset{o}{v}{}^{\sharp}(z) \to \overset{o}{v}{}^{\infty}(z)$ as $n \to \infty$ *pointwise*, but not uniformly, for $z \in \Sigma_{o}^{\sharp}$, whence, one can not conclude that $\overset{o}{\mathcal{M}}{}^{\sharp}(z) \to \overset{o}{m}{}^{\infty}(z)$ as $n \to \infty$ uniformly for $z \in \Sigma_{o}^{\sharp}$. The resolution of this lack of uniformity at the end-points of the support of the 'odd' equilibrium measure constitutes, therefore, the essential analytical obstacle remaining for the analysis of the RHP $(\overset{o}{\mathcal{M}}{}^{\sharp}(z), \overset{o}{v}{}^{\sharp}(z), \Sigma_{o}^{\sharp})$, and a substantial part of the following analysis is devoted to overcoming this problem.

The key necessary to remedy (and control) the above-mentioned analytical difficulty is to construct parametrices for the solution of the RHP $(\overset{o}{\mathcal{M}}{}^{\sharp}(z), \overset{o}{v}{}^{\sharp}(z), \Sigma_{o}^{\sharp})$ in 'small' neighbourhoods (open discs) about $\{b_{j-1}^{o}, a_{j}^{o}\}_{j=1}^{N+1}$ (where the convergence of $\overset{o}{v}{}^{\sharp}(z)$ to $\overset{o}{v}{}^{\infty}(z)$ as $n \to \infty$ is not uniform) in such a way that, on the boundary of these neighbourhoods, the parametrices 'match' with the solution of the model RHP, $\overset{o}{m}{}^{\infty}(z)$, up to $o(1)$ (in fact, $\mathcal{O}((n+1/2)^{-1})$) as $n \to \infty$; furthermore, in the generic framework considered in this work, namely, $\widetilde{V}\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is regular, in which case the (density of the) 'odd' equilibrium measure behaves as a square root at the end-points of $\mathrm{supp}(\mu_{V}^{o})$, that is, $\psi_{V}^{o}(s) =_{s \downarrow b_{j-1}^{o}} \mathcal{O}((s-b_{j-1}^{o})^{1/2})$ and $\psi_{V}^{o}(s) =_{s \uparrow a_{j}^{o}} \mathcal{O}((a_{j}^{o}-s)^{1/2})$, $j=1,\dots,N+1$, it is well known [3,47,79] that the parametrices can be expressed in terms of Airy functions. (The general method used to construct such parametrices is via a Vanishing Lemma [87].) More precisely, one surrounds the end-points of the support of the 'odd' equilibrium measure, $\{b_{j-1}^{o}, a_{j}^{o}\}_{j=1}^{N+1}$, by 'small', mutually disjoint (open) discs,

$$\mathbb{D}_{\epsilon}(b_{j-1}^{o}) := \left\{z \in \mathbb{C};\ |z-b_{j-1}^{o}| < \epsilon_{j}^{b}\right\} \qquad \text{and} \qquad \mathbb{D}_{\epsilon}(a_{j}^{o}) := \left\{z \in \mathbb{C};\ |z-a_{j}^{o}| < \epsilon_{j}^{a}\right\}, \quad j=1,\dots,N+1,$$

where $\epsilon_{j}^{b}, \epsilon_{j}^{a}$ are arbitrarily fixed, sufficiently small positive real numbers chosen so that $\mathbb{D}_{\epsilon}(b_{i-1}^{o}) \cap \mathbb{D}_{\epsilon}(a_{j}^{o}) = \emptyset$, $i,j = 1,\dots,N+1$, and defines $S_{p}^{o}(z)$, the parametrix for $\overset{o}{\mathcal{M}}{}^{\sharp}(z)$, by $\overset{o}{m}{}^{\infty}(z)$ for $z \in \mathbb{C} \setminus (\cup_{j=1}^{N+1}(\mathbb{D}_{\epsilon}(b_{j-1}^{o}) \cup \mathbb{D}_{\epsilon}(a_{j}^{o})))$, and by $m_{p}^{o}(z)$ for $z \in \cup_{j=1}^{N+1}(\mathbb{D}_{\epsilon}(b_{j-1}^{o}) \cup \mathbb{D}_{\epsilon}(a_{j}^{o}))$, and solves the local RHP for $m_{p}^{o}(z)$ on $\cup_{j=1}^{N+1}(\mathbb{D}_{\epsilon}(b_{j-1}^{o}) \cup \mathbb{D}_{\epsilon}(a_{j}^{o}))$ in such a way ('optimal', in the nomenclature of [47]) that $m_{p}^{o}(z) \approx_{n\to\infty} \overset{o}{m}{}^{\infty}(z)$ (to $\mathcal{O}((n+1/2)^{-1})$) for $z \in \cup_{j=1}^{N+1}(\partial\mathbb{D}_{\epsilon}(b_{j-1}^{o}) \cup \partial\mathbb{D}_{\epsilon}(a_{j}^{o}))$, whence $\mathcal{R}^{o}(z) := \overset{o}{\mathcal{M}}{}^{\sharp}(z)(S_{p}^{o}(z))^{-1}\colon \mathbb{C} \setminus \widetilde{\Sigma}_{o}^{\sharp} \to \mathrm{SL}_{2}(\mathbb{C})$, where $\widetilde{\Sigma}_{o}^{\sharp} := \Sigma_{o}^{\sharp} \cup (\cup_{j=1}^{N+1}(\partial\mathbb{D}_{\epsilon}(b_{j-1}^{o}) \cup \partial\mathbb{D}_{\epsilon}(a_{j}^{o})))$, solves the RHP $(\mathcal{R}^{o}(z), v_{\mathcal{R}}^{o}(z), \widetilde{\Sigma}_{o}^{\sharp})$ with $\|v_{\mathcal{R}}^{o}(\cdot)-\mathrm{I}\|_{\cap_{p\in\{2,\infty\}}\mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{p}(\widetilde{\Sigma}_{o}^{\sharp})} =_{n\to\infty} \mathcal{O}((n+1/2)^{-1})$ uniformly; in particular, the error term, which is $\mathcal{O}((n+1/2)^{-1})$ as $n \to \infty$, is uniform in $\cap_{p\in\{1,2,\infty\}}\mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{p}(\widetilde{\Sigma}_{o}^{\sharp})$. By general Riemann-Hilbert techniques (see, for example, [86]), $\mathcal{R}^{o}(z)$ (and thus $\overset{o}{\mathcal{M}}{}^{\sharp}(z)$ via the relation $\overset{o}{\mathcal{M}}{}^{\sharp}(z) = \mathcal{R}^{o}(z)S_{p}^{o}(z)$) can be computed to any order of $(n+1/2)^{-1}$ (as $n \to \infty$) via a Neumann series expansion (of the corresponding resolvent kernel). In fact, at the very core of the above-mentioned discussion, and the analysis that follows, is the following Corollary (see, for example, [79], Corollary 7.108):

**Corollary 4.1 (Deift [79]).** *For an oriented contour $\Sigma \subset \mathbb{C}$, let $m^{\infty}\colon \mathbb{C} \setminus \Sigma \to \mathrm{SL}_{2}(\mathbb{C})$ and $m^{(n)}\colon \mathbb{C} \setminus \Sigma \to \mathrm{SL}_{2}(\mathbb{C})$, $n \in \mathbb{N}$, respectively, solve the following, equivalent* RHPs, *$(m^{\infty}(z), v^{\infty}(z), \Sigma)$ and $(m^{(n)}(z), v^{(n)}(z), \Sigma)$, where*

$$v^{\infty}\colon \Sigma \to \mathrm{GL}_{2}(\mathbb{C}),\ z \mapsto (\mathrm{I}-w_{-}^{\infty}(z))^{-1}(\mathrm{I}+w_{+}^{\infty}(z))$$

*and*

$$v^{(n)}\colon \Sigma \to \mathrm{GL}_{2}(\mathbb{C}),\ z \mapsto \left(\mathrm{I}-w_{-}^{(n)}(z)\right)^{-1}\left(\mathrm{I}+w_{+}^{(n)}(z)\right),$$

*and suppose that $(\mathbf{id}-C_{w^{\infty}}^{\infty})^{-1}$ exists, where*

$$\mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{2}(\Sigma) \ni f \mapsto C_{w^{\infty}}^{\infty} f := C_{+}^{\infty}(fw_{-}^{\infty}) + C_{-}^{\infty}(fw_{+}^{\infty}),$$

*with*

$$C_{\pm}^{\infty}\colon \mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{2}(\Sigma) \to \mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{2}(\Sigma),\ f \mapsto (C_{\pm}^{\infty}f)(z) := \lim_{\substack{z' \to z \\ z' \in \pm\,\mathrm{side\,of}\,\Sigma}} \int_{\Sigma} \frac{f(s)}{s-z'}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},$$

*and $\|w_{l}^{(n)}(\cdot)-w_{l}^{\infty}(\cdot)\|_{\cap_{p\in\{2,\infty\}}\mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{p}(\Sigma)} \to 0$ as $n \to \infty$, $l=\pm1$. Then, $\exists\ N^{*} \in \mathbb{N}$ such that, $\forall\ n > N^{*}$, $m^{\infty}(z)$ and $m^{(n)}(z)$ exist, and $\|m_{l}^{(n)}(\cdot)-m_{l}^{\infty}(\cdot)\|_{\mathcal{L}_{\mathrm{M}_{2}(\mathbb{C})}^{2}(\Sigma)} \to 0$ as $n \to \infty$, $l=\pm1$.*

A detailed exposition, including further motivations, for the construction of parametrices of the above-mentioned type can be found in [3, 45–47, 49, 79]; rather than regurgitating, verbatim, much of the analysis that can be found in the latter references, the point of view taken here is that one follows the scheme presented therein to obtain the results stated below, that is, the parametrix for the RHP $(\overset{o}{\mathcal{M}}^{\sharp}(z), \overset{o}{\upsilon}^{\sharp}(z), \Sigma_o^{\sharp})$ formulated in Lemma 4.2. In the case of the right-most end-points of the support of the 'odd' equilibrium measure, $\{a_j^o\}_{j=1}^{N+1}$, a terse sketch of a proof is presented for the reader's convenience, and the remaining (left-most) end-points, namely, $b_0^o, b_1^o, \dots, b_N^o$, are analysed analogously.

The parametrix for the RHP $(\overset{o}{\mathcal{M}}^{\sharp}(z), \overset{o}{\upsilon}^{\sharp}(z), \Sigma_o^{\sharp})$ is now presented. By a parametrix of the RHP $(\overset{o}{\mathcal{M}}^{\sharp}(z), \overset{o}{\upsilon}^{\sharp}(z), \Sigma_o^{\sharp})$, in the neighbourhoods of the end-points of the support of the 'odd' equilibrium measure, $\{b_{j-1}^o, a_j^o\}_{j=1}^{N+1}$, is meant the solution of the RHPs formulated in the following two Lemmae (Lemmae 4.6 and 4.7). Define the 'small', mutually disjoint (open) discs about the end-points of the support of the 'odd' equilibrium measure as follows: $\mathbb{U}_{\delta_{b_{j-1}}}^o := \{z \in \mathbb{C};\, |z-b_{j-1}^o| < \delta_{b_{j-1}}^o \in (0,1)\}$ and $\mathbb{U}_{\delta_{a_j}}^o := \{z \in \mathbb{C};\, |z-a_j^o| < \delta_{a_j}^o \in (0,1)\}$, $j=1,\dots,N+1$, where $\delta_{b_{j-1}}^o$ and $\delta_{a_j}^o$ are sufficiently small, positive real numbers chosen (amongst other things: see Lemmae 4.6 and 4.7 below) so that $\mathbb{U}_{\delta_{b_{i-1}}}^o \cap \mathbb{U}_{\delta_{a_j}}^o = \varnothing$, $i,j=1,\dots,N+1$ (the corresponding regions $\Omega_{b_{j-1}}^{o,l}$ and $\Omega_{a_j}^{o,l}$, and arcs $\Sigma_{b_{j-1}}^{o,l}$ and $\Sigma_{a_j}^{o,l}$, $j=1,\dots,N+1$, $l=1,2,3,4$, respectively, are defined more precisely below; see, also, Figures 5 and 6).

**Remark 4.3.** In order to simplify the results of Lemmae 4.6 and 4.7 (see below), it is convenient to introduce the following notation: (i)

$$\Psi_1^o(z) := \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2\,\mathrm{Ai}'(\omega^2 z) \end{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}\pi}{6}\sigma_3}, \qquad \Psi_2^o(z) := \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2\,\mathrm{Ai}'(\omega^2 z) \end{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}\pi}{6}\sigma_3}(\mathrm{I}-\sigma_-),$$

$$\Psi_3^o(z) := \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2\,\mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}\pi}{6}\sigma_3}(\mathrm{I}+\sigma_-), \qquad \Psi_4^o(z) := \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2\,\mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}\pi}{6}\sigma_3},$$

where $\mathrm{Ai}(\cdot)$ is the Airy function (cf. Subsection 2.3), and $\omega = \exp(2\pi\mathrm{i}/3)$; and (ii)

$$\mho_j^o := \begin{cases} \Omega_j^o, & j=1,\dots,N, \\ 0, & j=0,N+1, \end{cases}$$

where $\Omega_j^o = 4\pi \int_{b_j^o}^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s$. ∎

**Lemma 4.6.** *Let* $\overset{o}{\mathcal{M}}^{\sharp} \colon \mathbb{C} \setminus \Sigma_o^{\sharp} \to \mathrm{SL}_2(\mathbb{C})$ *solve the* RHP $(\overset{o}{\mathcal{M}}^{\sharp}(z), \overset{o}{\upsilon}^{\sharp}(z), \Sigma_o^{\sharp})$ *formulated in Lemma* 4.2, *and set*

$$\mathbb{U}_{\delta_{b_{j-1}}}^o := \left\{ z \in \mathbb{C};\, |z-b_{j-1}^o| < \delta_{b_{j-1}}^o \in (0,1) \right\}, \quad j=1,\dots,N+1.$$

*Let*

$$\Phi_{b_{j-1}}^o(z) := \left( \frac{3}{4}\left(n+\frac{1}{2}\right) \xi_{b_{j-1}}^o(z) \right)^{2/3}, \quad j=1,\dots,N+1,$$

*with*

$$\xi_{b_{j-1}}^o(z) = -2 \int_z^{b_{j-1}^o} (R_o(s))^{1/2} h_V^o(s)\,\mathrm{d}s,$$

*where, for* $z \in \mathbb{U}_{\delta_{b_{j-1}}}^o \setminus (-\infty, b_{j-1}^o)$, $\xi_{b_{j-1}}^o(z) = \mathfrak{b}(z-b_{j-1}^o)^{3/2} G_{b_{j-1}}^o(z)$, *with* $\mathfrak{b} := \pm 1$ *for* $z \in \mathbb{C}_{\pm}$, *and* $G_{b_{j-1}}^o(z)$ *analytic, in particular,*

$$G_{b_{j-1}}^o(z) \underset{z \to b_{j-1}^o}{=} \frac{4}{3} f(b_{j-1}^o) + \frac{4}{5} f'(b_{j-1}^o)(z-b_{j-1}^o) + \frac{2}{7} f''(b_{j-1}^o)(z-b_{j-1}^o)^2 + \mathcal{O}\big((z-b_{j-1}^o)^3\big),$$

*where*

$$f(b_0^o) = \mathrm{i}(-1)^N h_V^o(b_0^o) \eta_{b_0^o},$$

$$f'(b_0^o)=\mathrm{i}(-1)^N\left(\frac{1}{2}h_V^o(b_0^o)\eta_{b_0^o}\left(\sum_{l=1}^N\left(\frac{1}{b_0^o-b_l^o}+\frac{1}{b_0^o-a_l^o}\right)+\frac{1}{b_0^o-a_{N+1}^o}\right)+(h_V^o(b_0^o))'\eta_{b_0^o}\right),$$

$$\begin{aligned}f''(b_0^o)&=\mathrm{i}(-1)^N\left(\frac{h_V^o(b_0^o)(h_V^o(b_0^o))''-((h_V^o(b_0^o))')^2}{h_V^o(b_0^o)}\eta_{b_0^o}-\frac{1}{2}h_V^o(b_0^o)\eta_{b_0^o}\right.\\&\times\left(\sum_{l=1}^N\left(\frac{1}{(b_0^o-b_l^o)^2}+\frac{1}{(b_0^o-a_l^o)^2}\right)+\frac{1}{(b_0^o-a_{N+1}^o)^2}\right)\\&+\left(\frac{1}{2}\left(\sum_{k=1}^N\left(\frac{1}{b_0^o-b_k^o}+\frac{1}{b_0^o-a_k^o}\right)+\frac{1}{b_0^o-a_{N+1}^o}\right)+\frac{(h_V^o(b_0^o))'}{h_V^o(b_0^o)}\right)\\&\times\left.\left(\frac{1}{2}h_V^o(b_0^o)\eta_{b_0^o}\left(\sum_{l=1}^N\left(\frac{1}{b_0^o-b_l^o}+\frac{1}{b_0^o-a_l^o}\right)+\frac{1}{b_0^o-a_{N+1}^o}\right)+(h_V^o(b_0^o))'\eta_{b_0^o}\right)\right),\end{aligned}$$

*with*

$$\eta_{b_0^o}:=\left((a_{N+1}^o-b_0^o)\prod_{k=1}^N(b_k^o-b_0^o)(a_k^o-b_0^o)\right)^{1/2}\qquad(>0),$$

*and, for* $j=1,\dots,N$,

$$\begin{aligned}f(b_j^o)&=\mathrm{i}(-1)^{N-j}h_V^o(b_j^o)\eta_{b_j^o},\\f'(b_j^o)&=\mathrm{i}(-1)^{N-j}\left(\frac{1}{2}h_V^o(b_j^o)\eta_{b_j^o}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{b_j^o-b_k^o}+\frac{1}{b_j^o-a_k^o}\right)+\frac{1}{b_j^o-a_j^o}+\frac{1}{b_j^o-a_{N+1}^o}+\frac{1}{b_j^o-b_0^o}\right)\right.\\&+\left.(h_V^o(b_j^o))'\eta_{b_j^o}\right),\\f''(b_j^o)&=\mathrm{i}(-1)^{N-j}\left(\frac{h_V^o(b_j^o)(h_V^o(b_j^o))''-((h_V^o(b_j^o))')^2}{h_V^o(b_j^o)}\eta_{b_j^o}-\frac{1}{2}h_V^o(b_j^o)\eta_{b_j^o}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{(b_j^o-b_k^o)^2}+\frac{1}{(b_j^o-a_k^o)^2}\right)\right.\right.\\&+\left.\frac{1}{(b_j^o-a_j^o)^2}+\frac{1}{(b_j^o-a_{N+1}^o)^2}+\frac{1}{(b_j^o-b_0^o)^2}\right)+\left(\frac{(h_V^o(b_j^o))'}{h_V^o(b_j^o)}+\frac{1}{2}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{b_j^o-b_k^o}+\frac{1}{b_j^o-a_k^o}\right)\right.\right.\\&+\left.\left.\frac{1}{b_j^o-a_j^o}+\frac{1}{b_j^o-a_{N+1}^o}+\frac{1}{b_j^o-b_0^o}\right)\right)\left(\frac{1}{2}h_V^o(b_j^o)\eta_{b_j^o}\left(\sum_{\substack{k=1\\k\neq j}}^N\left(\frac{1}{b_j^o-b_k^o}+\frac{1}{b_j^o-a_k^o}\right)\right.\right.\\&+\left.\left.\left.\frac{1}{b_j^o-a_j^o}+\frac{1}{b_j^o-a_{N+1}^o}+\frac{1}{b_j^o-b_0^o}\right)+(h_V^o(b_j^o))'\eta_{b_j^o}\right)\right),\end{aligned}$$

*with*

$$\eta_{b_j^o}:=\left((b_j^o-a_j^o)(a_{N+1}^o-b_j^o)(b_j^o-b_0^o)\prod_{k=1}^{j-1}(b_j^o-b_k^o)(b_j^o-a_k^o)\prod_{l=j+1}^N(b_l^o-b_j^o)(a_l^o-b_j^o)\right)^{1/2}\qquad(>0),$$

*and* $((0,1)\ni)\ \delta_{b_{j-1}}^o$, $j=1,\dots,N+1$, *are chosen sufficiently small so that* $\Phi_{b_{j-1}}^o(z)$, *which are bi-holomorphic, conformal, and non-orientation preserving, map* $\mathbb{U}_{\delta_{b_{j-1}}}^o$ *(and, thus, the oriented contours* $\Sigma_{b_{j-1}}^o:=\cup_{l=1}^4\Sigma_{b_{j-1}}^{o,l}$, $j=1,\dots,N+1$ *: Figure* 6*) injectively onto open* (*n-dependent*) *neighbourhoods* $\widehat{\mathbb{U}}_{\delta_{b_{j-1}}}^o$, $j=1,\dots,N+1$, *of* 0 *such that* $\Phi_{b_{j-1}}^o(b_{j-1}^o)=0$, $\Phi_{b_{j-1}}^o:\mathbb{U}_{\delta_{b_{j-1}}}^o\to\widehat{\mathbb{U}}_{\delta_{b_{j-1}}}^o:=\Phi_{b_{j-1}}^o(\mathbb{U}_{\delta_{b_{j-1}}}^o)$, $\Phi_{b_{j-1}}^o(\mathbb{U}_{\delta_{b_{j-1}}}^o\cap\Sigma_{b_{j-1}}^{o,l})=\Phi_{b_{j-1}}^o(\mathbb{U}_{\delta_{b_{j-1}}}^o)\cap\gamma_{b_{j-1}}^{o,l}$, *and* $\Phi_{b_{j-1}}^o(\mathbb{U}_{\delta_{b_{j-1}}}^o\cap\Omega_{b_{j-1}}^{o,l})=\Phi_{b_{j-1}}^o(\mathbb{U}_{\delta_{b_{j-1}}}^o)\cap\widehat{\Omega}_{b_{j-1}}^{o,l}$, $l=1,2,3,4$, *with* $\widehat{\Omega}_{b_{j-1}}^{o,1}=\{\zeta\in\mathbb{C};\ \arg(\zeta)\in(0,2\pi/3)\}$, $\widehat{\Omega}_{b_{j-1}}^{o,2}=\{\zeta\in\mathbb{C};\ \arg(\zeta)\in(2\pi/3,\pi)\}$, $\widehat{\Omega}_{b_{j-1}}^{o,3}=\{\zeta\in\mathbb{C};\ \arg(\zeta)\in(-\pi,-2\pi/3)\}$, *and* $\widehat{\Omega}_{b_{j-1}}^{o,4}=\{\zeta\in\mathbb{C};\ \arg(\zeta)\in(-2\pi/3,0)\}$.

*The parametrix for the* RHP $(\overset{o}{\mathcal{M}}{}^{\sharp}(z),\overset{o}{v}{}^{\sharp}(z),\Sigma_o^{\sharp})$, *for* $z\in\mathbb{U}^o_{\delta_{b_{j-1}}}$, $j=1,\dots,N+1$, *is the solution of the following* RHPs *for* $\mathcal{X}^o\colon\mathbb{U}^o_{\delta_{b_{j-1}}}\setminus\Sigma^o_{b_{j-1}}\to\mathrm{SL}_2(\mathbb{C})$, $j=1,\dots,N+1$, *where* $\Sigma^o_{b_{j-1}}:=(\Phi^o_{b_{j-1}})^{-1}(\gamma^o_{b_{j-1}})$, *with* $(\Phi^o_{b_{j-1}})^{-1}$ *denoting the inverse mapping, and* $\gamma^o_{b_{j-1}}:=\cup_{l=1}^4\gamma^{o,l}_{b_{j-1}}$: (i) $\mathcal{X}^o(z)$ *is holomorphic for* $z\in\mathbb{U}^o_{\delta_{b_{j-1}}}\setminus\Sigma^o_{b_{j-1}}$, $j=1,\dots,N+1$; (ii) $\mathcal{X}^o_{\pm}(z):=\lim_{\substack{z'\to z\\ z'\in\pm\,\text{side of}\,\Sigma^o_{b_{j-1}}}}\mathcal{X}^o(z')$, $j=1,\dots,N+1$, *satisfy the boundary condition*

$$\mathcal{X}^o_+(z)=\mathcal{X}^o_-(z)\overset{o}{v}{}^{\sharp}(z),\quad z\in\mathbb{U}^o_{\delta_{b_{j-1}}}\cap\Sigma^o_{b_{j-1}},\quad j=1,\dots,N+1,$$

*where* $\overset{o}{v}{}^{\sharp}(z)$ *is given in Lemma* 4.2; *and* (iii) *uniformly for* $z\in\partial\mathbb{U}^o_{\delta_{b_{j-1}}}:=\left\{z\in\mathbb{C};\,|z-b^o_{j-1}|=\delta^o_{b_{j-1}}\right\}$, $j=1,\dots,N+1$,

$$\overset{o}{m}{}^{\infty}(z)(\mathcal{X}^o(z))^{-1}\underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}^o_{\delta_{b_{j-1}}}}}{=}\mathrm{I}+\mathcal{O}((n+1/2)^{-1}),\quad j=1,\dots,N+1.$$

*The solutions of the* RHPs $(\mathcal{X}^o(z),\overset{o}{v}{}^{\sharp}(z),\mathbb{U}^o_{\delta_{b_{j-1}}}\cap\Sigma^o_{b_{j-1}})$, $j=1,\dots,N+1$, *are:*

**(1)** *for* $z\in\Omega^{o,1}_{b_{j-1}}:=\mathbb{U}^o_{\delta_{b_{j-1}}}\cap(\Phi^o_{b_{j-1}})^{-1}(\widehat{\Omega}^{o,1}_{b_{j-1}})$, $j=1,\dots,N+1$,

$$\mathcal{X}^o(z)=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}\overset{o}{m}{}^{\infty}(z)\sigma_3\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^o_{j-1}\,\mathrm{ad}(\sigma_3)}\begin{pmatrix}\mathrm{i}&-\mathrm{i}\\1&1\end{pmatrix}(\Phi^o_{b_{j-1}}(z))^{\frac{1}{4}\sigma_3}\Psi^o_1(\Phi^o_{b_{j-1}}(z))\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{b_{j-1}}(z)\sigma_3}\sigma_3,$$

*where* $\overset{o}{m}{}^{\infty}(z)$ *is given in Lemma* 4.5, *and* $\Psi^o_1(z)$ *and* $\mho^o_k$ *are defined in Remark* 4.4;

**(2)** *for* $z\in\Omega^{o,2}_{b_{j-1}}:=\mathbb{U}^o_{\delta_{b_{j-1}}}\cap(\Phi^o_{b_{j-1}})^{-1}(\widehat{\Omega}^{o,2}_{b_{j-1}})$, $j=1,\dots,N+1$,

$$\mathcal{X}^o(z)=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}\overset{o}{m}{}^{\infty}(z)\sigma_3\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^o_{j-1}\,\mathrm{ad}(\sigma_3)}\begin{pmatrix}\mathrm{i}&-\mathrm{i}\\1&1\end{pmatrix}(\Phi^o_{b_{j-1}}(z))^{\frac{1}{4}\sigma_3}\Psi^o_2(\Phi^o_{b_{j-1}}(z))\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{b_{j-1}}(z)\sigma_3}\sigma_3,$$

*where* $\Psi^o_2(z)$ *is defined in Remark* 4.4;

**(3)** *for* $z\in\Omega^{o,3}_{b_{j-1}}:=\mathbb{U}^o_{\delta_{b_{j-1}}}\cap(\Phi^o_{b_{j-1}})^{-1}(\widehat{\Omega}^{o,3}_{b_{j-1}})$, $j=1,\dots,N+1$,

$$\mathcal{X}^o(z)=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}\overset{o}{m}{}^{\infty}(z)\sigma_3\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^o_{j-1}\,\mathrm{ad}(\sigma_3)}\begin{pmatrix}\mathrm{i}&-\mathrm{i}\\1&1\end{pmatrix}(\Phi^o_{b_{j-1}}(z))^{\frac{1}{4}\sigma_3}\Psi^o_3(\Phi^o_{b_{j-1}}(z))\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{b_{j-1}}(z)\sigma_3}\sigma_3,$$

*where* $\Psi^o_3(z)$ *is defined in Remark* 4.4;

**(4)** *for* $z\in\Omega^{o,4}_{b_{j-1}}:=\mathbb{U}^o_{\delta_{b_{j-1}}}\cap(\Phi^o_{b_{j-1}})^{-1}(\widehat{\Omega}^{o,4}_{b_{j-1}})$, $j=1,\dots,N+1$,

$$\mathcal{X}^o(z)=\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}\overset{o}{m}{}^{\infty}(z)\sigma_3\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^o_{j-1}\,\mathrm{ad}(\sigma_3)}\begin{pmatrix}\mathrm{i}&-\mathrm{i}\\1&1\end{pmatrix}(\Phi^o_{b_{j-1}}(z))^{\frac{1}{4}\sigma_3}\Psi^o_4(\Phi^o_{b_{j-1}}(z))\mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^o_{b_{j-1}}(z)\sigma_3}\sigma_3,$$

*where* $\Psi^o_4(z)$ *is defined in Remark* 4.4.

**Lemma 4.7.** *Let* $\overset{o}{\mathcal{M}}{}^{\sharp}\colon\mathbb{C}\setminus\Sigma^{\sharp}_o\to\mathrm{SL}_2(\mathbb{C})$ *solve the* RHP $(\overset{o}{\mathcal{M}}{}^{\sharp}(z),\overset{o}{v}{}^{\sharp}(z),\Sigma^{\sharp}_o)$ *formulated in Lemma* 4.2, *and set*

$$\mathbb{U}^o_{\delta_{a_j}}:=\left\{z\in\mathbb{C};\,|z-a^o_j|<\delta^o_{a_j}\in(0,1)\right\},\quad j=1,\dots,N+1.$$

*Let*

$$\Phi^o_{a_j}(z):=\left(\frac{3}{4}\left(n+\frac{1}{2}\right)\xi^o_{a_j}(z)\right)^{2/3},\quad j=1,\dots,N+1,$$

*with*

$$\xi^o_{a_j}(z)=2\int_{a^o_j}^{z}(R_o(s))^{1/2}h^o_V(s)\,\mathrm{d}s,$$

*where, for* $z\in\mathbb{U}^o_{\delta_{a_j}}\setminus(-\infty,a^o_j)$, $\xi^o_{a_j}(z)=(z-a^o_j)^{3/2}G^o_{a_j}(z)$, $j=1,\dots,N+1$, *with* $G^o_{a_j}(z)$ *analytic, in particular,*

$$G^o_{a_j}(z)\underset{z\to a^o_j}{=}\frac{4}{3}f(a^o_j)+\frac{4}{5}f'(a^o_j)(z-a^o_j)+\frac{2}{7}f''(a^o_j)(z-a^o_j)^2+\mathcal{O}\Big((z-a^o_j)^3\Big),$$

*where*

$$
\begin{aligned}
f(a_{N+1}^{o}) &= h_{V}^{o}(a_{N+1}^{o})\eta_{a_{N+1}^{o}}, \\
f'(a_{N+1}^{o}) &= \frac{1}{2}h_{V}^{o}(a_{N+1}^{o})\eta_{a_{N+1}^{o}}\left(\sum_{l=1}^{N}\left(\frac{1}{a_{N+1}^{o}-b_{l}^{o}}+\frac{1}{a_{N+1}^{o}-a_{l}^{o}}\right)+\frac{1}{a_{N+1}^{o}-b_{0}^{o}}\right) \\
&\quad + (h_{V}^{o}(a_{N+1}^{o}))'\eta_{a_{N+1}^{o}}, \\
f''(a_{N+1}^{o}) &= \frac{h_{V}^{o}(a_{N+1}^{o})(h_{V}^{o}(a_{N+1}^{o}))''-((h_{V}^{o}(a_{N+1}^{o}))')^{2}}{h_{V}^{o}(a_{N+1}^{o})}\eta_{a_{N+1}^{o}}-\frac{1}{2}h_{V}^{o}(a_{N+1}^{o})\eta_{a_{N+1}^{o}} \\
&\quad \times\left(\sum_{l=1}^{N}\left(\frac{1}{(a_{N+1}^{o}-b_{l}^{o})^{2}}+\frac{1}{(a_{N+1}^{o}-a_{l}^{o})^{2}}\right)+\frac{1}{(a_{N+1}^{o}-b_{0}^{o})^{2}}\right) \\
&\quad +\left(\frac{1}{2}\left(\sum_{k=1}^{N}\left(\frac{1}{a_{N+1}^{o}-b_{k}^{o}}+\frac{1}{a_{N+1}^{o}-a_{k}^{o}}\right)+\frac{1}{a_{N+1}^{o}-b_{0}^{o}}\right)+\frac{(h_{V}^{o}(a_{N+1}^{o}))'}{h_{V}^{o}(a_{N+1}^{o})}\right) \\
&\quad \times\left(\frac{1}{2}h_{V}^{o}(a_{N+1}^{o})\eta_{a_{N+1}^{o}}\left(\sum_{l=1}^{N}\left(\frac{1}{a_{N+1}^{o}-a_{l}^{o}}+\frac{1}{a_{N+1}^{o}-b_{l}^{o}}\right)+\frac{1}{a_{N+1}^{o}-b_{0}^{o}}\right)\right. \\
&\quad \left. +\ (h_{V}^{o}(a_{N+1}^{o}))'\eta_{a_{N+1}^{o}}\right),
\end{aligned}
$$

*with*

$$
\eta_{a_{N+1}^{o}} := \left((a_{N+1}^{o}-b_{0}^{o})\prod_{k=1}^{N}(a_{N+1}^{o}-b_{k}^{o})(a_{N+1}^{o}-a_{k}^{o})\right)^{1/2} \qquad (>0),
$$

*and, for* $j=1,\dots,N$,

$$
\begin{aligned}
f(a_{j}^{o}) &= (-1)^{N-j+1}h_{V}^{o}(a_{j}^{o})\eta_{a_{j}^{o}}, \\
f'(a_{j}^{o}) &= (-1)^{N-j+1}\left(\frac{1}{2}h_{V}^{o}(a_{j}^{o})\eta_{a_{j}^{o}}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{a_{j}^{o}-b_{k}^{o}}+\frac{1}{a_{j}^{o}-a_{k}^{o}}\right)+\frac{1}{a_{j}^{o}-b_{j}^{o}}+\frac{1}{a_{j}^{o}-a_{N+1}^{o}}+\frac{1}{a_{j}^{o}-b_{0}^{o}}\right)\right. \\
&\quad \left. +\ (h_{V}^{o}(a_{j}^{o}))'\eta_{a_{j}^{o}}\right), \\
f''(a_{j}^{o}) &= (-1)^{N-j+1}\left(\frac{h_{V}^{o}(a_{j}^{o})(h_{V}^{o}(a_{j}^{o}))''-((h_{V}^{o}(a_{j}^{o}))')^{2}}{h_{V}^{o}(a_{j}^{o})}\eta_{a_{j}^{o}}-\frac{1}{2}h_{V}^{o}(a_{j}^{o})\eta_{a_{j}^{o}}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{(a_{j}^{o}-b_{k}^{o})^{2}}+\frac{1}{(a_{j}^{o}-a_{k}^{o})^{2}}\right)\right.\right. \\
&\quad \left. +\ \frac{1}{(a_{j}^{o}-b_{j}^{o})^{2}}+\frac{1}{(a_{j}^{o}-a_{N+1}^{o})^{2}}+\frac{1}{(a_{j}^{o}-b_{0}^{o})^{2}}\right)+\left(\frac{(h_{V}^{o}(a_{j}^{o}))'}{h_{V}^{o}(a_{j}^{o})}+\frac{1}{2}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{a_{j}^{o}-b_{k}^{o}}+\frac{1}{a_{j}^{o}-a_{k}^{o}}\right)\right.\right. \\
&\quad \left.\left. +\ \frac{1}{a_{j}^{o}-b_{j}^{o}}+\frac{1}{a_{j}^{o}-a_{N+1}^{o}}+\frac{1}{a_{j}^{o}-b_{0}^{o}}\right)\right)\left(\frac{1}{2}h_{V}^{o}(a_{j}^{o})\eta_{a_{j}^{o}}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{a_{j}^{o}-b_{k}^{o}}+\frac{1}{a_{j}^{o}-a_{k}^{o}}\right)\right.\right. \\
&\quad \left.\left.\left. +\ \frac{1}{a_{j}^{o}-b_{j}^{o}}+\frac{1}{a_{j}^{o}-a_{N+1}^{o}}+\frac{1}{a_{j}^{o}-b_{0}^{o}}\right)+(h_{V}^{o}(a_{j}^{o}))'\eta_{a_{j}^{o}}\right)\right),
\end{aligned}
$$

*with*

$$
\eta_{a_{j}^{o}} := \left((b_{j}^{o}-a_{j}^{o})(a_{N+1}^{o}-a_{j}^{o})(a_{j}^{o}-b_{0}^{o})\prod_{k=1}^{j-1}(a_{j}^{o}-b_{k}^{o})(a_{j}^{o}-a_{k}^{o})\prod_{l=j+1}^{N}(b_{l}^{o}-a_{j}^{o})(a_{l}^{o}-a_{j}^{o})\right)^{1/2} \qquad (>0),
$$

*and* $((0,1)\ni)\ \delta_{a_{j}}^{o}$, $j=1,\dots,N+1$, *are chosen sufficiently small so that* $\Phi_{a_{j}}^{o}(z)$, *which are bi-holomorphic, conformal, and orientation preserving, map* $\mathbb{U}_{\delta_{a_{j}}}^{o}$ *(and, thus, the oriented contours* $\Sigma_{a_{j}}^{o} := \cup_{l=1}^{4}\Sigma_{a_{j}}^{o,l}$, $j=1,\dots,N+1$ : *Figure* 5)

*injectively onto open (n-dependent) neighbourhoods* $\widehat{\mathbb{U}}^{o}_{\delta_{a_j}}$, $j=1,\dots,N+1$, *of* $0$ *such that* $\Phi^{o}_{a_j}(a^{o}_{j})=0$, $\Phi^{o}_{a_j}\colon \mathbb{U}^{o}_{\delta_{a_j}} \to \widehat{\mathbb{U}}^{o}_{\delta_{a_j}} := \Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}})$, $\Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}} \cap \Sigma^{o,l}_{a_j}) = \Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}}) \cap \gamma^{o,l}_{a_j}$, *and* $\Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}} \cap \Omega^{o,l}_{a_j}) = \Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}}) \cap \widehat{\Omega}^{o,l}_{a_j}$, $l=1,2,3,4$, *with* $\widehat{\Omega}^{o,1}_{a_j} = \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (0,2\pi/3)\}$, $\widehat{\Omega}^{o,2}_{a_j} = \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (2\pi/3,\pi)\}$, $\widehat{\Omega}^{o,3}_{a_j} = \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (-\pi,-2\pi/3)\}$, *and* $\widehat{\Omega}^{o,4}_{a_j} = \{\zeta \in \mathbb{C};\ \arg(\zeta) \in (-2\pi/3,0)\}$.

*The parametrix for the* RHP $(\overset{o}{\mathcal{M}}{}^{\sharp}(z), \overset{o}{v}{}^{\sharp}(z), \Sigma^{\sharp}_{o})$, *for* $z \in \mathbb{U}^{o}_{\delta_{a_j}}$, $j=1,\dots,N+1$, *is the solution of the following* RHPs *for* $\mathcal{X}^{o}\colon \mathbb{U}^{o}_{\delta_{a_j}} \setminus \Sigma^{o}_{a_j} \to \mathrm{SL}_2(\mathbb{C})$, $j=1,\dots,N+1$, *where* $\Sigma^{o}_{a_j} := (\Phi^{o}_{a_j})^{-1}(\gamma^{o}_{a_j})$, *with* $(\Phi^{o}_{a_j})^{-1}$ *denoting the inverse mapping, and* $\gamma^{o}_{a_j} := \cup_{l=1}^{4} \gamma^{o,l}_{a_j}$ : (i) $\mathcal{X}^{o}(z)$ *is holomorphic for* $z \in \mathbb{U}^{o}_{\delta_{a_j}} \setminus \Sigma^{o}_{a_j}$, $j=1,\dots,N+1$; (ii) $\mathcal{X}^{o}_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \Sigma^{o}_{a_j}}} \mathcal{X}^{o}(z')$, $j=1,\dots,N+1$, *satisfy the boundary condition*

$$\mathcal{X}^{o}_{+}(z) = \mathcal{X}^{o}_{-}(z)\overset{o}{v}{}^{\sharp}(z), \quad z \in \mathbb{U}^{o}_{\delta_{a_j}} \cap \Sigma^{o}_{a_j}, \quad j=1,\dots,N+1,$$

*where* $\overset{o}{v}{}^{\sharp}(z)$ *is given in Lemma* 4.2; *and* (iii) *uniformly for* $z \in \partial\mathbb{U}^{o}_{\delta_{a_j}} := \left\{z \in \mathbb{C};\ |z-a^{o}_{j}| = \delta^{o}_{a_j}\right\}$, $j=1,\dots,N+1$,

$$\overset{o}{m}{}^{\infty}(z)(\mathcal{X}^{o}(z))^{-1} \underset{\substack{n\to\infty \\ z \in \partial\mathbb{U}^{o}_{\delta_{a_j}}}}{=} \mathrm{I} + \mathcal{O}((n+1/2)^{-1}), \quad j=1,\dots,N+1.$$

*The solutions of the* RHPs $(\mathcal{X}^{o}(z), \overset{o}{v}{}^{\sharp}(z), \mathbb{U}^{o}_{\delta_{a_j}} \cap \Sigma^{o}_{a_j})$, $j=1,\dots,N+1$, *are:*

**(1)** *for* $z \in \Omega^{o,1}_{a_j} := \mathbb{U}^{o}_{\delta_{a_j}} \cap (\Phi^{o}_{a_j})^{-1}(\widehat{\Omega}^{o,1}_{a_j})$, $j=1,\dots,N+1$,

$$\mathcal{X}^{o}(z) = \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}} \overset{o}{m}{}^{\infty}(z) \mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^{o}_{j}\,\mathrm{ad}(\sigma_3)} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} (\Phi^{o}_{a_j}(z))^{\frac{1}{4}\sigma_3} \Psi^{o}_{1}(\Phi^{o}_{a_j}(z)) \mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_j}(z)\sigma_3},$$

*where* $\overset{o}{m}{}^{\infty}(z)$ *is given in Lemma* 4.5, *and* $\Psi^{o}_{1}(z)$ *and* $\mho^{o}_{k}$ *are defined in Remark* 4.4;

**(2)** *for* $z \in \Omega^{o,2}_{a_j} := \mathbb{U}^{o}_{\delta_{a_j}} \cap (\Phi^{o}_{a_j})^{-1}(\widehat{\Omega}^{o,2}_{a_j})$, $j=1,\dots,N+1$,

$$\mathcal{X}^{o}(z) = \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}} \overset{o}{m}{}^{\infty}(z) \mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^{o}_{j}\,\mathrm{ad}(\sigma_3)} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} (\Phi^{o}_{a_j}(z))^{\frac{1}{4}\sigma_3} \Psi^{o}_{2}(\Phi^{o}_{a_j}(z)) \mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_j}(z)\sigma_3},$$

*where* $\Psi^{o}_{2}(z)$ *is defined in Remark* 4.4;

**(3)** *for* $z \in \Omega^{o,3}_{a_j} := \mathbb{U}^{o}_{\delta_{a_j}} \cap (\Phi^{o}_{a_j})^{-1}(\widehat{\Omega}^{o,3}_{a_j})$, $j=1,\dots,N+1$,

$$\mathcal{X}^{o}(z) = \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}} \overset{o}{m}{}^{\infty}(z) \mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^{o}_{j}\,\mathrm{ad}(\sigma_3)} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} (\Phi^{o}_{a_j}(z))^{\frac{1}{4}\sigma_3} \Psi^{o}_{3}(\Phi^{o}_{a_j}(z)) \mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_j}(z)\sigma_3},$$

*where* $\Psi^{o}_{3}(z)$ *is defined in Remark* 4.4;

**(4)** *for* $z \in \Omega^{o,4}_{a_j} := \mathbb{U}^{o}_{\delta_{a_j}} \cap (\Phi^{o}_{a_j})^{-1}(\widehat{\Omega}^{o,4}_{a_j})$, $j=1,\dots,N+1$,

$$\mathcal{X}^{o}(z) = \sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}} \overset{o}{m}{}^{\infty}(z) \mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^{o}_{j}\,\mathrm{ad}(\sigma_3)} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} (\Phi^{o}_{a_j}(z))^{\frac{1}{4}\sigma_3} \Psi^{o}_{4}(\Phi^{o}_{a_j}(z)) \mathrm{e}^{\frac{1}{2}(n+\frac{1}{2})\xi^{o}_{a_j}(z)\sigma_3},$$

*where* $\Psi^{o}_{4}(z)$ *is defined in Remark* 4.4.

**Remark 4.4.** Perusing Lemmae 4.6 and 4.7, one notes that the normalisation condition at zero, which is needed in order to guarantee the existence of solutions to the corresponding (parametrix) RHPs, is absent. The normalisation conditions at zero are replaced by the (uniform) matching conditions $\overset{o}{m}{}^{\infty}(z)(\mathcal{X}^{o}(z))^{-1} =_{\substack{n\to\infty \\ z\in\partial\mathbb{U}^{o}_{\delta_{*_j}}}} \mathrm{I} + \mathcal{O}((n+1/2)^{-1})$, where $*_j \in \{b_{j-1}, a_j\}$, $j=1,\dots,N+1$, with $\partial\mathbb{U}^{o}_{\delta_{*_j}}$ defined in Lemmae 4.6 and 4.7. ■

*Sketch of proof of Lemma* 4.7. Let $(\overset{o}{\mathcal{M}}{}^{\sharp}(z), \overset{o}{v}{}^{\sharp}(z), \Sigma_o^{\sharp})$ be the RHP formulated in Lemma 4.2, and recall the definitions stated therein. For each $a_j^o \in \mathrm{supp}(\mu_V^o)$, $j=1,\dots,N+1$, define $\mathbb{U}_{\delta_{a_j}}^o$, $j=1,\dots,N+1$, as in the Lemma, that is, surround each right-most end-point $a_j^o$ by open discs of radius $\delta_{a_j}^o \in (0,1)$ centred at $a_j^o$. Recalling the formula for $\overset{o}{v}{}^{\sharp}(z)$ given in Lemma 4.2, one shows, via the proof of Lemma 4.1, that:

(1) $4\pi\mathrm{i}\int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s = 4\pi\mathrm{i}(\int_z^{a_j^o} + \int_{a_j^o}^{b_j^o} + \int_{b_j^o}^{a_{N+1}^o})\psi_V^o(s)\,\mathrm{d}s$, whence, recalling the expression for the density of the 'odd' equilibrium measure given in Lemma 3.5, that is, $\mathrm{d}\mu_V^o(x) := \psi_V^o(x)\,\mathrm{d}x = \frac{1}{2\pi\mathrm{i}}(R_o(x))_+^{1/2} h_V^o(x)\mathbf{1}_{J_o}(x)\,\mathrm{d}x$, one arrives at, upon considering the analytic continuation of $4\pi\mathrm{i} \cdot \int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s$ to $\mathbb{C} \setminus \mathbb{R}$ (cf. proof of Lemma 4.1), in particular, to the oriented (open) skeletons $\mathbb{U}_{\delta_{a_j}}^o \cap (J_j^{o,\frown} \cup J_j^{o,\smile})$, $j=1,\dots,N+1$, $4\pi\mathrm{i}\int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s = -\xi_{a_j}^o(z) + \mathrm{i}\mho_j^o$, $j=1,\dots,N+1$, where $\xi_{a_j}^o(z) = 2\int_{a_j^o}^z (R_o(s))^{1/2} h_V^o(s)\,\mathrm{d}s$, and $\mho_j^o$ are defined in Remark 4.4;

(2) $g_+^o(z) + g_-^o(z) - \widetilde{V}(z) - \ell_o - \mathfrak{Q}_{\mathcal{A}}^+ - \mathfrak{Q}_{\mathcal{A}}^- = -(2+\frac{1}{n})\int_{a_j^o}^z (R_o(s))^{1/2} h_V^o(s)\,\mathrm{d}s < 0$, $z \in (a_{N+1}^o, +\infty) \cup (\cup_{j=1}^N (a_j^o, b_j^o))$.

Via the latter formulae, which appear in the $(i\,j)$-elements, $i,j=1,2$, of the jump matrix $\overset{o}{v}{}^{\sharp}(z)$, denoting $\overset{o}{\mathcal{M}}{}^{\sharp}(z)$ by $\mathcal{X}^o(z)$ for $z \in \mathbb{U}_{\delta_{a_j}}^o$, $j=1,\dots,N+1$, and defining

$$\mathcal{P}_{a_j}^o(z) := \begin{cases} \mathcal{X}^o(z)\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi_{a_j}^o(z)\sigma_3}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o\sigma_3}, & z \in \mathbb{C}_+ \cap \mathbb{U}_{\delta_{a_j}}^o, \quad j=1,\dots,N+1, \\ \mathcal{X}^o(z)\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})\xi_{a_j}^o(z)\sigma_3}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o\sigma_3}, & z \in \mathbb{C}_- \cap \mathbb{U}_{\delta_{a_j}}^o, \quad j=1,\dots,N+1, \end{cases}$$

one notes that $\mathcal{P}_{a_j}^o \colon \mathbb{U}_{\delta_{a_j}}^o \setminus J_{a_j}^o \to \mathrm{GL}_2(\mathbb{C})$, where $J_{a_j}^o := J_j^{o,\frown} \cup J_j^{o,\smile} \cup (a_j^o - \delta_{a_j}^o, a_j^o + \delta_{a_j}^o)$, $j=1,\dots,N+1$, solves the RHP $(\mathcal{P}_{a_j}^o(z), v_{\mathcal{P}_{a_j}}^o(z), J_{a_j}^o)$, with constant jump matrices $v_{\mathcal{P}_{a_j}}^o(z)$, $j=1,\dots,N+1$, defined by

$$v_{\mathcal{P}_{a_j}}^o(z) := \begin{cases} \mathrm{I} + \sigma_-, & z \in \mathbb{U}_{\delta_{a_j}}^o \cap (J_j^{o,\frown} \cup J_j^{o,\smile}) = \Sigma_{a_j}^{o,1} \cup \Sigma_{a_j}^{o,3}, \\ \mathrm{I} + \sigma_+, & z \in \mathbb{U}_{\delta_{a_j}}^o \cap (a_j^o, a_j^o + \delta_{a_j}^o) = \Sigma_{a_j}^{o,4}, \\ \mathrm{i}\sigma_2, & z \in \mathbb{U}_{\delta_{a_j}}^o \cap (a_j^o - \delta_{a_j}^o, a_j^o) = \Sigma_{a_j}^{o,2}, \end{cases}$$

subject, still, to the asymptotic matching conditions $\overset{o}{m}{}^{\infty}(z)(\mathcal{X}^o(z))^{-1} =_{n\to\infty} \mathrm{I} + \mathcal{O}((n+1/2)^{-1})$, uniformly for $z \in \partial\mathbb{U}_{\delta_{a_j}}^o$, $j=1,\dots,N+1$.

Set, as in the Lemma, $\Phi_{a_j}^o(z) := (\frac{3}{4}(n+\frac{1}{2})\xi_{a_j}^o(z))^{2/3}$, $j=1,\dots,N+1$, with $\xi_{a_j}^o(z)$ defined above: a careful analysis of the branch cuts shows that, for $z \in \mathbb{U}_{\delta_{a_j}}^o$, $j=1,\dots,N+1$, $\Phi_{a_j}^o(z)$ and $\xi_{a_j}^o(z)$ satisfy the properties stated in the Lemma; in particular, for $\Phi_{a_j}^o \colon \mathbb{U}_{\delta_{a_j}}^o \to \mathbb{C}$, $j=1,\dots,N+1$, $\Phi_{a_j}^o(z) = (z-a_j^o)^{3/2} G_{a_j}^o(z)$, with $G_{a_j}^o(z)$ holomorphic for $z \in \mathbb{U}_{\delta_{a_j}}^o$ and characterised in the Lemma, $\Phi_{a_j}^o(a_j^o) = 0$, $(\Phi_{a_j}^o(z))' \neq 0$, $z \in \mathbb{U}_{\delta_{a_j}}^o$, and where $(\Phi_{a_j}^o(a_j^o))' = ((n+\frac{1}{2})f(a_j^o))^{2/3} > 0$, with $f(a_j^o)$ given in the Lemma. One now chooses $\delta_{a_j}^o$ $(\in (0,1))$, $j=1,\dots,N+1$, and the oriented—open—skeletons ('near' $a_j^o$) $J_{a_j}^o$, $j=1,\dots,N+1$, in such a way that their image under the bi-holomorphic, conformal and orientation-preserving mappings $\Phi_{a_j}^o(z)$ are the union of the straight-line segments $\gamma_{a_j}^{o,l}$, $l=1,2,3,4$, $j=1,\dots,N+1$. Set $\zeta := \Phi_{a_j}^o(z)$, $j=1,\dots,N+1$, and consider $\mathcal{X}^o(\Phi_{a_j}^o(z)) := \Psi^o(\zeta)$. Recalling the properties of $\Phi_{a_j}^o(z)$, a straightforward calculation shows that $\Psi^o \colon \Phi_{a_j}^o(\mathbb{U}_{\delta_{a_j}}^o) \setminus \cup_{l=1}^4 \gamma_{a_j}^{o,l} \to \mathrm{GL}_2(\mathbb{C})$, $j=1,\dots,N+1$, solves the RHPs $(\Psi^o(\zeta), v_{\Psi^o}^o(\zeta), \cup_{l=1}^4 \gamma_{a_j}^{o,l})$, $j=1,\dots,N+1$, with constant jump matrices $v_{\Psi^o}^o(\zeta)$, $j=1,\dots,N+1$, defined by

$$v_{\Psi^o}^o(\zeta) := \begin{cases} \mathrm{I} + \sigma_-, & \zeta \in \gamma_{a_j}^{o,1} \cup \gamma_{a_j}^{o,3}, \\ \mathrm{I} + \sigma_+, & \zeta \in \gamma_{a_j}^{o,4}, \\ \mathrm{i}\sigma_2, & \zeta \in \gamma_{a_j}^{o,2}. \end{cases}$$

The solution of the latter (yet-to-be normalised) RHPs is well known; in fact, their solution is expressed

in terms of the Airy function, and is given by (see, for example, [3, 46, 47, 49, 79])

$$\Psi^{o}(\zeta)=\begin{cases}\Psi^{o}_{1}(\zeta), & \zeta\in\widehat{\Omega}^{o,1}_{a_j}, \quad j=1,\dots,N+1,\\ \Psi^{o}_{2}(\zeta), & \zeta\in\widehat{\Omega}^{o,2}_{a_j}, \quad j=1,\dots,N+1,\\ \Psi^{o}_{3}(\zeta), & \zeta\in\widehat{\Omega}^{o,3}_{a_j}, \quad j=1,\dots,N+1,\\ \Psi^{o}_{4}(\zeta), & \zeta\in\widehat{\Omega}^{o,4}_{a_j}, \quad j=1,\dots,N+1,\end{cases}$$

where $\Psi^{o}_{k}(z)$, $k=1,2,3,4$, are defined in Remark 4.4. Recalling that $\Phi^{o}_{a_j}(z)$, $j=1,\dots,N+1$, are biholomorphic, and orientation-preserving conformal mappings, with $\Phi^{o}_{a_j}(a^{o}_{j})=0$ and $\Phi^{o}_{a_j}$ $(:\mathbb{U}^{o}_{\delta_{a_j}}\to\Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}})=:\widehat{\mathbb{U}}^{o}_{\delta_{a_j}})$ : $\mathbb{U}^{o}_{\delta_{a_j}}\cap J^{o}_{a_j}\to\Phi^{o}_{a_j}(\mathbb{U}^{o}_{\delta_{a_j}}\cap J^{o}_{a_j})=\widehat{\mathbb{U}}^{o}_{\delta_{a_j}}\cap(\cup_{l=1}^{4}\gamma^{o,l}_{a_j})$, $j=1,\dots,N+1$, one notes that, for any analytic maps $E^{o}_{a_j}\colon\mathbb{U}^{o}_{\delta_{a_j}}\to\mathrm{GL}_2(\mathbb{C})$, $j=1,\dots,N+1$, $\mathbb{U}^{o}_{\delta_{a_j}}\setminus J^{o}_{a_j}\ni\zeta\mapsto E^{o}_{a_j}(\zeta)\Psi^{o}(\zeta)$ also solves the latter RHPs $(\Psi^{o}(\zeta),\upsilon^{o}_{\Psi^{o}}(\zeta),\cup_{l=1}^{4}\gamma^{o,l}_{a_j})$, $j=1,\dots,N+1$: one uses this 'degree of freedom' of 'multiplying on the left' by a non-degenerate, analytic, matrix-valued function in order to satisfy the remaining asymptotic (as $n\to\infty$) matching condition for the parametrix, namely, $\overset{o}{m}{}^{\infty}(z)(\mathcal{X}^{o}(z))^{-1}\underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}^{o}_{\delta_{a_j}}}}{=}\mathrm{I}+\mathcal{O}((n+1/2)^{-1})$, uniformly for $z\in\partial\mathbb{U}^{o}_{\delta_{a_j}}$, $j=1,\dots,N+1$.

Consider, say, and without loss of generality, the regions $\Omega^{o,1}_{a_j}:=(\Phi^{o}_{a_j})^{-1}(\widehat{\Omega}^{o,1}_{a_j})$, $j=1,\dots,N+1$ (Figure 5). Re-tracing the above transformations, one shows that, for $z\in\Omega^{o,1}_{a_j}$ $(\subset\mathbb{C}_{+})$, $j=1,\dots,N+1$, $\mathcal{X}^{o}(z)=E^{o}_{a_j}(z)\Psi^{o}((\frac{3}{4}(n+\frac{1}{2})\xi^{o}_{a_j}(z))^{2/3})\exp(\frac{1}{2}(n+\frac{1}{2})(\xi^{o}_{a_j}(z)-\mathrm{i}\mho^{o}_{j})\sigma_3)$, whence, using the expression above for $\Psi^{o}(\zeta)$, $\zeta\in\mathbb{C}_{+}\cap\widehat{\Omega}^{o,1}_{a_j}$, $j=1,\dots,N+1$, and the asymptotic expansions for $\mathrm{Ai}(\cdot)$ and $\mathrm{Ai}'(\cdot)$ (as $n\to\infty$) given in Equations (2.6), one arrives at

$$\mathcal{X}^{o}(z)\underset{\substack{n\to\infty\\ z\in\partial\Omega^{o,1}_{a_j}\cap\partial\mathbb{U}^{o}_{\delta_{a_j}}}}{=}\frac{1}{\sqrt{2\pi}}E^{o}_{a_j}(z)\left(\left(\frac{3}{4}\left(n+\frac{1}{2}\right)\xi^{o}_{a_j}(z)\right)^{2/3}\right)^{-\frac{1}{4}\sigma_3}\begin{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}} & \mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\\ -\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}} & -\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\end{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^{o}_{j}\sigma_3}$$
$$\times\left(\mathrm{I}+\mathcal{O}\left((n+1/2)^{-1}\right)\right):$$

demanding that, for $z\in\partial\Omega^{o,1}_{a_j}\cap\partial\mathbb{U}^{o}_{\delta_{a_j}}$, $j=1,\dots,N+1$, $\overset{o}{m}{}^{\infty}(z)(\mathcal{X}^{o}(z))^{-1}=_{n\to\infty}\mathrm{I}+\mathcal{O}((n+1/2)^{-1})$, one gets that

$$E^{o}_{a_j}(z)=\frac{1}{\sqrt{2\mathrm{i}}}\overset{o}{m}{}^{\infty}(z)\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho^{o}_{j}\sigma_3}\begin{pmatrix}\mathrm{i} & -\mathrm{i}\\ 1 & 1\end{pmatrix}\left(\left(\frac{3}{4}\left(n+\frac{1}{2}\right)\xi^{o}_{a_j}(z)\right)^{2/3}\right)^{\frac{1}{4}\sigma_3}, \quad j=1,\dots,N+1$$

(note that $\det(E^{o}_{a_j}(z))=1$). One mimicks the above paradigm for the remaining boundary skeletons $\partial\Omega^{o,l}_{a_j}\cap\partial\mathbb{U}^{o}_{\delta_{a_j}}$, $l=2,3,4$, $j=1,\dots,N+1$, and shows that the exact same formula for $E^{o}_{a_j}(z)$ given above is obtained; thus, for $E^{o}_{a_j}(z)$, $j=1,\dots,N+1$, as given above, one concludes that, uniformly for $z\in\partial\mathbb{U}^{o}_{\delta_{a_j}}$, $j=1,\dots,N+1$, $\overset{o}{m}{}^{\infty}(z)(\mathcal{X}^{o}(z))^{-1}\underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}^{o}_{\delta_{a_j}}}}{=}\mathrm{I}+\mathcal{O}((n+1/2)^{-1})$. There remains, however, the question of unimodularity, since

$$\det(\mathcal{X}^{o}(z))=\begin{vmatrix}\mathrm{Ai}(\Phi^{o}_{a_j}(z)) & \mathrm{Ai}(\omega^2\Phi^{o}_{a_j}(z))\\ \mathrm{Ai}'(\Phi^{o}_{a_j}(z)) & \omega^2\,\mathrm{Ai}'(\omega^2\Phi^{o}_{a_j}(z))\end{vmatrix}\quad\text{or}\quad\begin{vmatrix}\mathrm{Ai}(\Phi^{o}_{a_j}(z)) & -\omega^2\,\mathrm{Ai}(\omega\Phi^{o}_{a_j}(z))\\ \mathrm{Ai}'(\Phi^{o}_{a_j}(z)) & -\mathrm{Ai}'(\omega\Phi^{o}_{a_j}(z))\end{vmatrix}:$$

multiplying $\mathcal{X}^{o}(z)$ on the left by a constant, $\widetilde{c}$, say, using the Wronskian relations (see Chapter 10 of [82]) $\mathrm{W}(\mathrm{Ai}(\lambda),\mathrm{Ai}(\omega^2\lambda))=(2\pi)^{-1}\exp(\mathrm{i}\pi/6)$ and $\mathrm{W}(\mathrm{Ai}(\lambda),\mathrm{Ai}(\omega\lambda))=-(2\pi)^{-1}\exp(-\mathrm{i}\pi/6)$, and the linear dependence relation for Airy functions, $\mathrm{Ai}(\lambda)+\omega\,\mathrm{Ai}(\omega\lambda)+\omega^2\,\mathrm{Ai}(\omega^2\lambda)=0$, one shows that, upon imposing the condition $\det(\mathcal{X}^{o}(z))=1$, $\widetilde{c}=(2\pi)^{1/2}\exp(-\mathrm{i}\pi/12)$. □

The above analyses lead one to the following lemma.

**Lemma 4.8.** *Let* $\overset{o}{\mathcal{M}}{}^{\sharp}\colon\mathbb{C}\setminus\Sigma^{\sharp}_{o}\to\mathrm{SL}_2(\mathbb{C})$ *solve the* RHP $(\overset{o}{\mathcal{M}}{}^{\sharp}(z),\overset{o}{\upsilon}{}^{\sharp}(z),\Sigma^{\sharp}_{o})$ *formulated in Lemma* 4.2*. Define*

$$\mathcal{S}^{o}_{p}(z):=\begin{cases}\overset{o}{m}{}^{\infty}(z), & z\in\mathbb{C}\setminus\cup_{j=1}^{N+1}(\mathbb{U}^{o}_{\delta_{b_{j-1}}}\cup\mathbb{U}^{o}_{\delta_{a_j}}),\\ \mathcal{X}^{o}(z), & z\in\cup_{j=1}^{N+1}(\mathbb{U}^{o}_{\delta_{b_{j-1}}}\cup\mathbb{U}^{o}_{\delta_{a_j}}),\end{cases}$$

*where* $\overset{o}{m}^{\infty} \colon \mathbb{C} \setminus J_o^{\infty} \to \mathrm{SL}_2(\mathbb{C})$ *is characterised completely in Lemma* 4.5, *and:* (1) *for* $z \in \mathbb{U}^o_{\delta_{b_{j-1}}}$, $j=1,\dots,N+1$, $\mathcal{X}^o \colon \mathbb{U}^o_{\delta_{b_{j-1}}} \setminus \Sigma^o_{b_{j-1}} \to \mathrm{SL}_2(\mathbb{C})$ *solve the* RHPs $(\mathcal{X}^o(z), \overset{o}{v}^{\sharp}(z), \Sigma^o_{b_{j-1}})$, $j=1,\dots,N+1$, *formulated in Lemma* 4.6; *and* (2) *for* $z \in \mathbb{U}^o_{\delta_{a_j}}$, $j=1,\dots,N+1$, $\mathcal{X}^o \colon \mathbb{U}^o_{\delta_{a_j}} \setminus \Sigma^o_{a_j} \to \mathrm{SL}_2(\mathbb{C})$ *solve the* RHPs $(\mathcal{X}^o(z), \overset{o}{v}^{\sharp}(z), \Sigma^o_{a_j})$, $j=1,\dots,N+1$, *formulated in Lemma* 4.7. *Set*

$$\mathcal{R}^o(z) := \overset{o}{\mathcal{M}}^{\sharp}(z)\big(\mathcal{S}^o_p(z)\big)^{-1},$$

*and define the augmented contour* $\Sigma^o_p := \Sigma^{\sharp}_o \cup (\cup_{j=1}^{N+1}(\partial \mathbb{U}^o_{\delta_{b_{j-1}}} \cup \mathbb{U}^o_{\delta_{a_j}}))$, *with the orientation given in Figure* 9. *Then* $\mathcal{R}^o \colon \mathbb{C} \setminus \Sigma^o_p \to \mathrm{SL}_2(\mathbb{C})$ *solves the following* RHP: (i) $\mathcal{R}^o(z)$ *is holomorphic for* $z \in \mathbb{C} \setminus \Sigma^o_p$; (ii) $\mathcal{R}^o_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \Sigma^o_p}} \mathcal{R}^o(z')$ *satisfy the boundary condition*

$$\mathcal{R}^o_+(z) = \mathcal{R}^o_-(z) v^o_{\mathcal{R}}(z), \quad z \in \Sigma^o_p,$$

*where*

$$v^o_{\mathcal{R}}(z) := \begin{cases} v^{o,1}_{\mathcal{R}}(z), & z \in (-\infty, b^o_0 - \delta^o_{b_0}) \cup (a^o_{N+1} + \delta^o_{a_{N+1}}, +\infty) =: \Sigma^{o,1}_p, \\ v^{o,2}_{\mathcal{R}}(z), & z \in (a^o_j + \delta^o_{a_j}, b^o_j - \delta^o_{b_j}) =: \Sigma^{o,2}_{p,j} \subset \cup_{l=1}^{N} \Sigma^{o,2}_{p,l} =: \Sigma^{o,2}_p, \\ v^{o,3}_{\mathcal{R}}(z), & z \in \cup_{j=1}^{N+1}(J^{o,\frown}_j \setminus (\mathbb{C}_+ \cap (\mathbb{U}^o_{\delta_{b_{j-1}}} \cup \mathbb{U}^o_{\delta_{a_j}}))) =: \Sigma^{o,3}_p, \\ v^{o,4}_{\mathcal{R}}(z), & z \in \cup_{j=1}^{N+1}(J^{o,\smile}_j \setminus (\mathbb{C}_- \cap (\mathbb{U}^o_{\delta_{b_{j-1}}} \cup \mathbb{U}^o_{\delta_{a_j}}))) =: \Sigma^{o,4}_p, \\ v^{o,5}_{\mathcal{R}}(z), & z \in \cup_{j=1}^{N+1}(\partial \mathbb{U}^o_{\delta_{b_{j-1}}} \cup \mathbb{U}^o_{\delta_{a_j}}) =: \Sigma^{o,5}_p, \\ \mathrm{I}, & z \in \Sigma^o_p \setminus \cup_{l=1}^{5} \Sigma^{o,l}_p, \end{cases}$$

*with*

$$\begin{aligned} v^{o,1}_{\mathcal{R}}(z) &= \mathrm{I} + \mathrm{e}^{n(g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^+_{\mathcal{A}}-\mathfrak{Q}^-_{\mathcal{A}})}\, \overset{o}{m}^{\infty}(z)\sigma_+(\overset{o}{m}^{\infty}(z))^{-1}, \\ v^{o,2}_{\mathcal{R}}(z) &= \mathrm{I} + \mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega^o_j+n(g^o_+(z)+g^o_-(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^+_{\mathcal{A}}-\mathfrak{Q}^-_{\mathcal{A}})}\, \overset{o}{m}^{\infty}_-(z)\sigma_+(\overset{o}{m}^{\infty}_-(z))^{-1}, \\ v^{o,3}_{\mathcal{R}}(z) &= \mathrm{I} + \mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\, \overset{o}{m}^{\infty}(z)\sigma_-(\overset{o}{m}^{\infty}(z))^{-1}, \\ v^{o,4}_{\mathcal{R}}(z) &= \mathrm{I} + \mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\, \overset{o}{m}^{\infty}(z)\sigma_-(\overset{o}{m}^{\infty}(z))^{-1}, \\ v^{o,5}_{\mathcal{R}}(z) &= \mathcal{X}^o(z)(\overset{o}{m}^{\infty}(z))^{-1}: \end{aligned}$$

(iii) $\mathcal{R}^o(z) \underset{\substack{z \to 0 \\ z \in \mathbb{C} \setminus \Sigma^o_p}}{=} \mathrm{I} + \mathcal{O}(z)$; *and* (iv) $\mathcal{R}^o(z) \underset{\substack{z \to \infty \\ z \in \mathbb{C} \setminus \Sigma^o_p}}{=} \mathcal{O}(1)$.

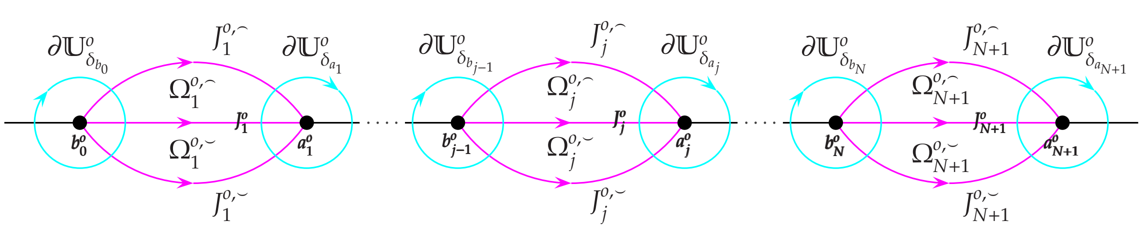


Figure 9: The augmented contour $\Sigma^o_p := \Sigma^{\sharp}_o \cup (\cup_{j=1}^{N+1}(\partial \mathbb{U}^o_{\delta_{b_{j-1}}} \cup \partial \mathbb{U}^o_{\delta_{a_j}}))$

*Proof.* Define the oriented, augmented skeleton $\Sigma^o_p$ as in the Lemma: the RHP $(\mathcal{R}^o(z), v^o_{\mathcal{R}}(z), \Sigma^o_p)$ follows from the RHPs $(\overset{o}{\mathcal{M}}^{\sharp}(z), \overset{o}{v}^{\sharp}(z), \Sigma^{\sharp}_o)$ and $(\overset{o}{m}^{\infty}(z), \overset{o}{v}^{\infty}(z), J^{\infty}_o)$ formulated in Lemmae 4.2 and 4.3, respectively, upon using the definitions of $\mathcal{S}^o_p(z)$ and $\mathcal{R}^o(z)$ given in the Lemma. □

# 5 Asymptotic (as $n\to\infty$) Solution of the RHP for $\overset{o}{\mathrm{Y}}(z)$

In this section, via the Beals-Coifman (BC) construction [74], the (normalised at zero) RHP $(\mathcal{R}^{o}(z), v^{o}_{\mathcal{R}}(z), \Sigma^{o}_{p})$ formulated in Lemma 4.8 is solved asymptotically (as $n\to\infty$); in particular, it is shown that, uniformly for $z\in\Sigma^{o}_{p}$,

$$\|v^{o}_{\mathcal{R}}(\cdot)-\mathrm{I}\|_{\cap_{p\in\{1,2,\infty\}}\mathcal{L}^{p}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{p})} \underset{n\to\infty}{=} \mathrm{I}+\mathcal{O}\Big(f(n)(n+1/2)^{-1}\Big),$$

where $f(n)=_{n\to\infty}\mathcal{O}(1)$, and, subsequently, the original **RHP2**, that is, $(\overset{o}{\mathrm{Y}}(z),\mathrm{I}+\mathrm{e}^{-n\widetilde{V}(z)}\sigma_{+},\mathbb{R})$, is solved asymptotically by re-tracing the finite sequence of RHP transformations $\mathcal{R}^{o}(z)$ (Lemmae 5.3 and 4.8) $\to\overset{o}{\mathcal{M}}{}^{\sharp}(z)$ (Lemma 4.2) $\to\overset{o}{\mathcal{M}}{}^{\flat}(z)$ (Proposition 4.1) $\to\overset{o}{\mathcal{M}}(z)$ (Lemma 3.4) $\to\overset{o}{\mathrm{Y}}(z)$. The (unique) solution for $\overset{o}{\mathrm{Y}}(z)$ then leads to the final asymptotic results for $z\pi_{2n+1}(z)$ (in the entire complex plane), $\xi^{(2n+1)}_{-n-1}$ and $\phi_{2n+1}(z)$ (in the entire complex plane) stated, respectively, in Theorems 2.3.1 and 2.3.2.

**Proposition 5.1.** *Let* $\mathcal{R}^{o}\colon\mathbb{C}\setminus\Sigma^{o}_{p}\to\mathrm{SL}_2(\mathbb{C})$ *solve the* RHP $(\mathcal{R}^{o}(z), v^{o}_{\mathcal{R}}(z), \Sigma^{o}_{p})$ *formulated in Lemma* 4.8. *Then:*

(1) *for* $z\in(-\infty,b^{o}_{0}-\delta^{o}_{b_0})\cup(a^{o}_{N+1}+\delta^{o}_{a_{N+1}},+\infty)=:\Sigma^{o,1}_{p}$,

$$v^{o}_{\mathcal{R}}(z)\underset{n\to\infty}{=}\begin{cases}\mathrm{I}+\mathcal{O}(f_{\infty}(n)\mathrm{e}^{-(n+\frac{1}{2})c_{\infty}|z|}), & z\in\Sigma^{o,1}_{p}\setminus\mathbb{U}^{o}_{0},\\ \mathrm{I}+\mathcal{O}(f_{0}(n)\mathrm{e}^{-(n+\frac{1}{2})c_{0}|z|^{-1}}), & z\in\Sigma^{o}_{p}\cap\mathbb{U}^{o}_{0},\end{cases}$$

*where* $c_0,c_{\infty}>0$, $(f_{\infty}(n))_{ij}=_{n\to\infty}\mathcal{O}(1)$, $(f_{0}(n))_{ij}=_{n\to\infty}\mathcal{O}(1)$, $i,j=1,2$, *and* $\mathbb{U}^{o}_{0}:=\{z\in\mathbb{C};\,|z|<\epsilon\}$, *with* $\epsilon$ *some arbitrarily fixed, sufficiently small positive real number;*

(2) *for* $z\in(a^{o}_{j}+\delta^{o}_{a_j},b^{o}_{j}-\delta^{o}_{b_j})=:\Sigma^{o,2}_{p,j}\subset\cup_{l=1}^{N}\Sigma^{o,2}_{p,l}=:\Sigma^{o,2}_{p}$, $j=1,\dots,N$,

$$v^{o}_{\mathcal{R}}(z)\underset{n\to\infty}{=}\begin{cases}\mathrm{I}+\mathcal{O}(f_{j}(n)\mathrm{e}^{-(n+\frac{1}{2})c_{j}(z-a^{o}_{j})}), & z\in\Sigma^{o,2}_{p,j}\setminus\mathbb{U}^{o}_{0},\\ \mathrm{I}+\mathcal{O}(\widetilde{f}_{j}(n)\mathrm{e}^{-(n+\frac{1}{2})\widetilde{c}_{j}|z|^{-1}}), & z\in\Sigma^{o,2}_{p,j}\cap\mathbb{U}^{o}_{0},\end{cases}$$

*where* $c_j,\widetilde{c}_j>0$, $(f_j(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, *and* $(\widetilde{f}_j(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$;

(3) *for* $z\in\cup_{j=1}^{N+1}(J^{o,\frown}_{j}\setminus(\mathbb{C}_{+}\cap(\mathbb{U}^{o}_{\delta_{b_{j-1}}}\cup\mathbb{U}^{o}_{\delta_{a_j}})))=:\Sigma^{o,3}_{p}$,

$$v^{o}_{\mathcal{R}}(z)\underset{n\to\infty}{=}\mathrm{I}+\mathcal{O}(\widehat{f}(n)\mathrm{e}^{-(n+\frac{1}{2})\widehat{c}|z|}),$$

*where* $\widehat{c}>0$ *and* $(\widehat{f}(n))_{ij}=_{n\to\infty}\mathcal{O}(1)$, $i,j=1,2$;

(4) *for* $z\in\cup_{j=1}^{N+1}(J^{o,\smile}_{j}\setminus(\mathbb{C}_{-}\cap(\mathbb{U}^{o}_{\delta_{b_{j-1}}}\cup\mathbb{U}^{o}_{\delta_{a_j}})))=:\Sigma^{o,4}_{p}$,

$$v^{o}_{\mathcal{R}}(z)\underset{n\to\infty}{=}\mathrm{I}+\mathcal{O}(\breve{f}(n)\mathrm{e}^{-(n+\frac{1}{2})\breve{c}|z|}),$$

*where* $\breve{c}>0$ *and* $(\breve{f}(n))_{ij}=_{n\to\infty}\mathcal{O}(1)$, $i,j=1,2$; *and*

(5) *for* $z\in\cup_{j=1}^{N+1}(\partial\mathbb{U}^{o}_{\delta_{b_{j-1}}}\cup\partial\mathbb{U}^{o}_{\delta_{a_j}})=:\Sigma^{o,5}_{p}$, *with* $j=1,\dots,N+1$,

$$v^{o}_{\mathcal{R}}(z)\underset{\substack{n\to\infty\\ z\in\mathbb{C}_{\pm}\cap\partial\mathbb{U}^{o}_{\delta_{b_{j-1}}}}}{=}\mathrm{I}+\frac{1}{(n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z)}\overset{o}{\mathfrak{M}}{}^{\infty}(z)\begin{pmatrix}\mp(s_1+t_1) & \mp\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}}\\ \mp\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j-1}} & \pm(s_1+t_1)\end{pmatrix}$$
$$\times(\overset{o}{\mathfrak{M}}{}^{\infty}(z))^{-1}+\mathcal{O}\left(\frac{1}{((n+\frac{1}{2})\xi^{o}_{b_{j-1}}(z))^{2}}\overset{o}{\mathfrak{M}}{}^{\infty}(z)f^{o}_{b_{j-1}}(n)(\overset{o}{\mathfrak{M}}{}^{\infty}(z))^{-1}\right),$$

*where* $\overset{o}{\mathfrak{M}}{}^{\infty}(z)$ *is characterised completely in Lemma* 4.5, $s_1=5/72$, $t_1=-7/72$, *for* $j=1,\dots,N+1$, $\xi^{o}_{b_{j-1}}(z)=-2\int_{z}^{b^{o}_{j-1}}(R_o(s))^{1/2}h^{o}_{V}(s)\,\mathrm{d}s=(z-b^{o}_{j-1})^{3/2}G^{o}_{b_{j-1}}(z)$, *with* $G^{o}_{b_{j-1}}(z)$ *described completely in Lemma* 4.6, $\mho^{o}_{j-1}$ *is defined in Remark* 4.4, *and* $(f^{o}_{b_{j-1}}(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$, *and*

$$v^{o}_{\mathcal{R}}(z)\underset{\substack{n\to\infty\\ z\in\mathbb{C}_{\pm}\cap\partial\mathbb{U}^{o}_{\delta_{a_j}}}}{=}\mathrm{I}+\frac{1}{(n+\frac{1}{2})\xi^{o}_{a_j}(z)}\overset{o}{\mathfrak{M}}{}^{\infty}(z)\begin{pmatrix}\mp(s_1+t_1) & \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}}\\ \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\mho^{o}_{j}} & \pm(s_1+t_1)\end{pmatrix}$$

$$\times(\overset{o}{\mathfrak{M}}^{\infty}(z))^{-1}+\mathcal{O}\left(\frac{1}{((n+\frac{1}{2})\xi^{o}_{a_j}(z))^{2}}\overset{o}{\mathfrak{M}}^{\infty}(z)f^{o}_{a_j}(n)(\overset{o}{\mathfrak{M}}^{\infty}(z))^{-1}\right), \quad j=1,\dots,N+1,$$

*where, for* $j=1,\dots,N+1$, $\xi^{o}_{a_j}(z)=2\int_{a^{o}_{j}}^{z}(R_o(s))^{1/2}h^{o}_{V}(s)\,\mathrm{d}s=(z-a^{o}_{j})^{3/2}G^{o}_{a_j}(z)$, *with* $G^{o}_{a_j}(z)$ *described completely in Lemma* 4.7, *and* $(f^{o}_{a_j}(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$.

*Proof.* Recall the definition of $v^{o}_{\mathcal{R}}(z)$ given in Lemma 4.8. For $z\in\Sigma^{o,1}_{p}:=(-\infty,b^{o}_{0}-\delta^{o}_{b_0})\cup(a^{o}_{N+1}+\delta^{o}_{a_{N+1}},+\infty)$, recall from Lemma 4.8 that

$$v^{o}_{\mathcal{R}}(z):=v^{o,1}_{\mathcal{R}}(z)=\mathrm{I}+\exp\Big(n(g^{o}_{+}(z)+g^{o}_{-}(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^{+}_{\mathcal{A}}-\mathfrak{Q}^{-}_{\mathcal{A}})\Big)\overset{o}{m}^{\infty}(z)\sigma_{+}(\overset{o}{m}^{\infty}(z))^{-1},$$

and, from the proof of Lemma 4.1, $g^{o}_{+}(z)+g^{o}_{-}(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^{+}_{\mathcal{A}}-\mathfrak{Q}^{-}_{\mathcal{A}}$ equals $-(2+\frac{1}{n})\int_{a^{o}_{N+1}}^{z}(R_o(s))^{1/2}\cdot h^{o}_{V}(s)\,\mathrm{d}s$ $(<0)$ for $z\in(a^{o}_{N+1}+\delta^{o}_{a_{N+1}},+\infty)$ and equals $(2+\frac{1}{n})\int_{z}^{b^{o}_{0}}(R_o(s))^{1/2}h^{o}_{V}(s)\,\mathrm{d}s$ $(<0)$ for $z\in(-\infty,b^{o}_{0}-\delta^{o}_{b_0})$; hence, recalling that $\widetilde{V}\colon\mathbb{R}\setminus\{0\}\to\mathbb{R}$, which is regular, satisfies conditions (2.3)–(2.5), using the asymptotic expansions (as $|z|\to\infty$ and $|z|\to 0$) for $g^{o}_{+}(z)+g^{o}_{-}(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^{+}_{\mathcal{A}}-\mathfrak{Q}^{-}_{\mathcal{A}}$ given in the proof of Lemma 3.6, that is, $g^{o}_{+}(z)+g^{o}_{-}(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^{+}_{\mathcal{A}}-\mathfrak{Q}^{-}_{\mathcal{A}}=_{|z|\to\infty}(1+\frac{1}{n})\ln(z^{2}+1)-\widetilde{V}(z)+\mathcal{O}(1)$ and $g^{o}_{+}(z)+g^{o}_{-}(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^{+}_{\mathcal{A}}-\mathfrak{Q}^{-}_{\mathcal{A}}=_{|z|\to 0}\ln(z^{-2}+1)-\widetilde{V}(z)+\mathcal{O}(1)$, upon recalling the expression for $\overset{o}{m}^{\infty}(z)$ given in Lemma 4.5 and noting that the respective factors $(\gamma^{o}(0))^{-1}\gamma^{o}(z)\pm\gamma^{o}(0)(\gamma^{o}(z))^{-1}$ and $\boldsymbol{\theta}^{o}(\pm\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}\pm\boldsymbol{d}_o)$ are uniformly bounded (with respect to $z$) in compact subsets outside the open intervals surrounding the end-points of the suppport of the 'odd' equilibrium measure, defining $\mathbb{U}^{o}_{0}$ as in the Proposition, one arrives at the asymptotic (as $n\to\infty$) estimates for $v^{o}_{\mathcal{R}}(z)$ on $\Sigma^{o,1}_{p}\setminus\mathbb{U}^{o}_{0}\ni z$ and $\Sigma^{o,1}_{p}\cap\mathbb{U}^{o}_{0}\ni z$ stated in item (1) of the Proposition. (It should be noted that the $n$-dependence of the $\mathrm{GL}_2(\mathbb{C})$-valued factors $f_{\infty}(n)$ and $f_0(n)$ are inherited from the bounded ($\mathcal{O}(1)$) $n$-dependence of the respective Riemann theta functions, whose corresponding series converge absolutely and uniformly due to the fact that the associated Riemann matrix of $\boldsymbol{\beta}^{o}$-periods, $\tau^{o}$, is pure imaginary and $-\mathrm{i}\tau^{o}$ is positive definite.)

For $z\in\Sigma^{o,2}_{p,j}:=(a^{o}_{j}+\delta^{o}_{a_j},b^{o}_{j}-\delta^{o}_{b_j})$, $j=1,\dots,N$, recall from Lemma 4.8 that

$$v^{o}_{\mathcal{R}}(z):=v^{o,2}_{\mathcal{R}}(z)=\mathrm{I}+\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega^{o}_{j}}\exp\Big(n(g^{o}_{+}(z)+g^{o}_{-}(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^{+}_{\mathcal{A}}-\mathfrak{Q}^{-}_{\mathcal{A}})\Big)\overset{o}{m}^{\infty}_{-}(z)\sigma_{+}(\overset{o}{m}^{\infty}_{-}(z))^{-1},$$

and, from the proof of Lemma 4.1, $g^{o}_{+}(z)+g^{o}_{-}(z)-\widetilde{V}(z)-\ell_o-\mathfrak{Q}^{+}_{\mathcal{A}}-\mathfrak{Q}^{-}_{\mathcal{A}}=-(2+\frac{1}{n})\int_{a^{o}_{j}}^{z}(R_o(s))^{1/2}h^{o}_{V}(s)\,\mathrm{d}s$ $(<0)$. Recalling, also, that $(R_o(z))^{1/2}:=(\prod_{k=1}^{N+1}(z-b^{o}_{k-1})(z-a^{o}_{k}))^{1/2}$ is continuous (and bounded) on the compact intervals $[a^{o}_{j},b^{o}_{j}]\supset\Sigma^{o,2}_{p,j}\ni z$, $j=1,\dots,N$, vanishes at the end-points $\{a^{o}_{j}\}_{j=1}^{N}$ (resp., $\{b^{o}_{j}\}_{j=1}^{N}$) like $(R_o(z))^{1/2}=_{z\downarrow a^{o}_{j}}\mathcal{O}((z-a^{o}_{j})^{1/2})$ (resp., $(R_o(z))^{1/2}=_{z\uparrow b^{o}_{j}}\mathcal{O}((z-b^{o}_{j})^{1/2})$), and is differentiable on the open intervals $\Sigma^{o,2}_{p,j}\ni z$, and $h^{o}_{V}(z)=\frac{1}{2}(2+\frac{1}{n})^{-1}\oint_{C^{o}_{\mathrm{R}}}(\frac{2\mathrm{i}}{\pi s}+\frac{\mathrm{i}\widetilde{V}'(s)}{\pi})(R_o(s))^{-1/2}(s-z)^{-1}\,\mathrm{d}s$ is analytic, it follows that, for $z\in\Sigma^{o,2}_{p,j}$,

$$\inf_{z\in\Sigma^{o,2}_{p,j}}(R_o(z))^{1/2}=:\widehat{m}_j\leqslant(R_o(z))^{1/2}\leqslant\widehat{M}_j:=\sup_{z\in\Sigma^{o,2}_{p,j}}(R_o(z))^{1/2}, \quad j=1,\dots,N;$$

thus, recalling the expression for $\overset{o}{m}^{\infty}(z)$ given in Lemma 4.5 and noting that the respective factors $(\gamma^{o}(0))^{-1}\gamma^{o}(z)\pm\gamma^{o}(0)(\gamma^{o}(z))^{-1}$ and $\boldsymbol{\theta}^{o}(\pm\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}\pm\boldsymbol{d}_o)$ are uniformly bounded (with respect to $z$) in compact subsets outside the open intervals surrounding the end-points of the suppport of the 'odd' equilibrium measure, and defining $\mathbb{U}^{o}_{0}$ as in the Proposition, after a straightforward integration argument, one arrives at the asymptotic (as $n\to\infty$) estimates for $v^{o}_{\mathcal{R}}(z)$ on $\Sigma^{o,2}_{p,j}\setminus\mathbb{U}^{o}_{0}\ni z$ and $\Sigma^{o,2}_{p,j}\cap\mathbb{U}^{o}_{0}\ni z$, $j=1,\dots,N$, stated in item (2) of the Proposition (the $n$-dependence of the $\mathrm{GL}_2(\mathbb{C})$-valued factors $f_j(n)$, $\widetilde{f_j}(n)$, $j=1,\dots,N$, is inherited from the bounded ($\mathcal{O}(1)$) $n$-dependence of the respective Riemann theta functions).

For $z\in\Sigma^{o,3}_{p}:=\cup_{j=1}^{N+1}(J^{o,\frown}_{j}\setminus(\mathbb{C}_{+}\cap(\mathbb{U}^{o}_{\delta_{b_{j-1}}}\cup\mathbb{U}^{o}_{\delta_{a_j}})))$, recall from Lemma 4.1 that $\mathrm{Re}(\mathrm{i}\int_{z}^{a^{o}_{N+1}}\psi^{o}_{V}(s)\,\mathrm{d}s)>0$ for $z\in\mathbb{C}_{+}\cap(\cup_{j=1}^{N+1}\mathbb{U}^{o}_{j})\supset\Sigma^{o,3}_{p}$, where $\mathbb{U}^{o}_{j}:=\{z\in\mathbb{C}^{*};\,\mathrm{Re}(z)\in(b^{o}_{j-1},a^{o}_{j}),\,\inf_{q\in(b^{o}_{j-1},a^{o}_{j})}|z-q|<r_j\in(0,1)\}$,

$j=1,\dots,N+1$, with $\mathbb{U}_i^o \cap \mathbb{U}_j^o = \varnothing$, $i \neq j=1,\dots,N+1$, and, from the proof of Lemma 4.8,

$$v_{\mathcal{R}}^o(z) := v_{\mathcal{R}}^{o,3}(z) = \mathrm{I} + \exp\left(-4\left(n+\frac{1}{2}\right)\pi\mathrm{i}\int_z^{a_{N+1}^o} \psi_V^o(s)\,\mathrm{d}s\right)\overset{o}{m}^{\infty}(z)\sigma_-(\overset{o}{m}^{\infty}(z))^{-1}:$$

using the expression for $\overset{o}{m}^{\infty}(z)$ given in Lemma 4.5 and noting that the respective factors $(\gamma^o(0))^{-1} \cdot \gamma^o(z) \pm \gamma^o(0)(\gamma^o(z))^{-1}$ and $\boldsymbol{\theta}^o(\pm\boldsymbol{u}^o(z) - \frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o \pm \boldsymbol{d}_o)$ are uniformly bounded (with respect to $z$) in compact subsets outside the open intervals surrounding the end-points of the suppport of the 'odd' equilibrium measure, an arc-length-parametrisation argument, complemented by an application of the Maximum Length (ML) Theorem, leads one directly to the asymptotic (as $n \to \infty$) estimate for $v_{\mathcal{R}}^o(z)$ on $\Sigma_p^{o,3} \ni z$ stated in item (3) of the Proposition (the $n$-dependence of the $\mathrm{GL}_2(\mathbb{C})$-valued factor $\widehat{f}(n)$ is inherited from the bounded ($\mathcal{O}(1)$) $n$-dependence of the respective Riemann theta functions). The above argument applies, *mutatis mutandis*, for the asymptotic estimate of $v_{\mathcal{R}}^o(z)$ on $\Sigma_p^{o,4} := \cup_{j=1}^{N+1}(J_j^{o,\smile} \setminus (\mathbb{C}_- \cap (\mathbb{U}_{\delta_{b_{j-1}}}^o \cup \mathbb{U}_{\delta_{a_j}}^o))) \ni z$ stated in item (4) of the Proposition.

Since the estimates in item (5) of the Proposition are similar, consider, say, and without loss of generality, the asymptotic (as $n \to \infty$) estimate for $v_{\mathcal{R}}^o(z)$ on $\partial\mathbb{U}_{\delta_{a_j}}^o \ni z$, $j=1,\dots,N+1$: this argument applies, *mutatis mutandis*, for the large-$n$ asymptotics of $v_{\mathcal{R}}^o(z)$ on $\cup_{j=1}^{N+1}\partial\mathbb{U}_{\delta_{b_{j-1}}}^o \ni z$. For $z \in \partial\mathbb{U}_{\delta_{a_j}}^o$, $j=1,\dots,N+1$, recall from the proof of Lemma 4.8 that $v_{\mathcal{R}}^o(z) := v_{\mathcal{R}}^{o,5}(z) = \mathcal{X}^o(z)(\overset{o}{m}^{\infty}(z))^{-1}$: using the expression for the parametrix, $\mathcal{X}^o(z)$, given in Lemma 4.7, and the large-argument asymptotics for the Airy function and its derivative given in Equations (2.6), one shows that, for $z \in \mathbb{C}_+ \cap \partial\mathbb{U}_{\delta_{a_j}}^o$, $j=1,\dots,N+1$,

$$\begin{aligned} v_{\mathcal{R}}^o(z) \underset{n\to\infty}{=} \mathrm{I} &+ \frac{\mathrm{e}^{-\mathrm{i}\pi/3}}{(n+\frac{1}{2})\xi_{a_j}^o(z)}\overset{o}{m}^{\infty}(z)\begin{pmatrix} \mathrm{i}\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} & -\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} \\ \mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} & \mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} \end{pmatrix}\begin{pmatrix} -s_1\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} & s_1\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} \\ t_1\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} & -t_1\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} \end{pmatrix} \\ &\times(\overset{o}{m}^{\infty}(z))^{-1} + \mathcal{O}\left(\frac{1}{((n+\frac{1}{2})\xi_{a_j}^o(z))^2}\overset{o}{m}^{\infty}(z)\begin{pmatrix} * & * \\ * & * \end{pmatrix}(\overset{o}{m}^{\infty}(z))^{-1}\right), \end{aligned}$$

where $\xi_{a_j}^o(z)$ and $\mho_j^o$, $j=1,\dots,N+1$, and $s_1$ and $t_1$ are defined in the Proposition, $\overset{o}{m}^{\infty}(z)$ is given in Lemma 4.5, and $\left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{C})$, and, for $z \in \mathbb{C}_- \cap \partial\mathbb{U}_{\delta_{a_j}}^o$, $j=1,\dots,N+1$,

$$\begin{aligned} v_{\mathcal{R}}^o(z) \underset{n\to\infty}{=} \mathrm{I} &+ \frac{\mathrm{e}^{-\mathrm{i}\pi/3}}{(n+\frac{1}{2})\xi_{a_j}^o(z)}\overset{o}{m}^{\infty}(z)\begin{pmatrix} \mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} & -\mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} \\ \mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} & \mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} \end{pmatrix}\begin{pmatrix} -s_1\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} & s_1\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} \\ t_1\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}\mathrm{e}^{\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} & -t_1\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{\mathrm{i}}{2}(n+\frac{1}{2})\mho_j^o} \end{pmatrix} \\ &\times(\overset{o}{m}^{\infty}(z))^{-1} + \mathcal{O}\left(\frac{1}{((n+\frac{1}{2})\xi_{a_j}^o(z))^2}\overset{o}{m}^{\infty}(z)\begin{pmatrix} * & * \\ * & * \end{pmatrix}(\overset{o}{m}^{\infty}(z))^{-1}\right). \end{aligned}$$

Upon recalling the formula for $\overset{o}{m}^{\infty}(z)$ in terms of $\overset{o}{\mathfrak{M}}^{\infty}(z)$ given in Lemma 4.5, and noting that the respective factors $(\gamma^o(0))^{-1}\gamma^o(z) \pm \gamma^o(0)(\gamma^o(z))^{-1}$ and $\boldsymbol{\theta}^o(\pm\boldsymbol{u}^o(z) - \frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o \pm \boldsymbol{d}_o)$ are uniformly bounded (with respect to $z$) in compact subsets outside the open intervals surrounding the end-points of the suppport of the 'odd' equilibrium measure, after a straightforward matrix-multiplication argument, one arrives at the asymptotic (as $n \to \infty$) estimates for $v_{\mathcal{R}}^o(z)$ on $\partial\mathbb{U}_{\delta_{a_j}}^o \ni z$, $j=1,\dots,N+1$, stated in item (5) of the Proposition (the $n$-dependence of the $\mathrm{GL}_2(\mathbb{C})$-valued factors $f_{a_j}^o(n)$, $j=1,\dots,N+1$, is inherited from the bounded ($\mathcal{O}(1)$) $n$-dependence of the respective Riemann theta functions). □

**Definition 5.1.** *For an oriented contour $D \subset \mathbb{C}$, let $\mathcal{N}_q(D)$ denote the set of all bounded linear operators from $\mathcal{L}^q_{\mathrm{M}_2(\mathbb{C})}(D)$ into $\mathcal{L}^q_{\mathrm{M}_2(\mathbb{C})}(D)$, $q \in \{1,2,\infty\}$.*

Since the analysis that follows relies substantially on the BC [74] construction for the solution of a matrix (and suitably normalised) RHP on an oriented and unbounded contour, it is convenient to present, with some requisite preamble, a succinct and self-contained synopsis of it at this juncture. One agrees to call a contour $\Gamma^\sharp$ *oriented* if:

(1) $\mathbb{C} \setminus \Gamma^\sharp$ has finitely many open connected components;

(2) $\mathbb{C} \setminus \Gamma^{\sharp}$ is the disjoint union of two, possibly disconnected, open regions, denoted by $\mho^{+}$ and $\mho^{-}$;
(3) $\Gamma^{\sharp}$ may be viewed as either the positively oriented boundary for $\mho^{+}$ or the negatively oriented boundary for $\mho^{-}$ ($\mathbb{C} \setminus \Gamma^{\sharp}$ is coloured by two colours, $\pm$).

Let $\Gamma^{\sharp}$, as a closed set, be the union of finitely many oriented, simple, piecewise-smooth arcs. Denote the set of all self-intersections of $\Gamma^{\sharp}$ by $\widehat{\Gamma}^{\sharp}$ (with $\mathrm{card}(\widehat{\Gamma}^{\sharp}) < \infty$ assumed throughout). Set $\widetilde{\Gamma}^{\sharp} := \Gamma^{\sharp} \setminus \widehat{\Gamma}^{\sharp}$. The BC [74] construction for the solution of a (matrix) RHP, in the absence of a discrete spectrum and spectral singularities [88] (see, also, [75,76,89–91]), on an oriented contour $\Gamma^{\sharp}$ consists of finding function $\mathcal{Y}(z) \colon \mathbb{C} \setminus \Gamma^{\sharp} \to \mathrm{M}_2(\mathbb{C})$ such that:

(1) $\mathcal{Y}(z)$ is holomorphic for $z \in \mathbb{C} \setminus \Gamma^{\sharp}$, $\mathcal{Y}(z) \restriction_{\mathbb{C}\setminus\Gamma^{\sharp}}$ has a continuous extension (from 'above' and 'below') to $\widetilde{\Gamma}^{\sharp}$, and $\lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \widetilde{\Gamma}^{\sharp}}} \int_{\widetilde{\Gamma}^{\sharp}} |\mathcal{Y}(z') - \mathcal{Y}_{\pm}(z)|^2 \, |\mathrm{d}z| = 0$;
(2) $\mathcal{Y}_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \widetilde{\Gamma}^{\sharp}}} \mathcal{Y}(z')$ satisfy $\mathcal{Y}_{+}(z) = \mathcal{Y}_{-}(z) v(z)$, $z \in \widetilde{\Gamma}^{\sharp}$, for some (smooth) 'jump' matrix $v \colon \widetilde{\Gamma}^{\sharp} \to \mathrm{GL}_2(\mathbb{C})$; and
(3) for arbitrarily fixed $\lambda_o \in \mathbb{C}$, and uniformly with respect to $z$, $\mathcal{Y}(z) \underset{\substack{z \to \lambda_o \\ z \in \mathbb{C}\setminus\Gamma^{\sharp}}}{=} \mathrm{I} + o(1)$, where $o(1) = \mathcal{O}(z - \lambda_o)$ if $\lambda_o$ is finite, and $o(1) = \mathcal{O}(z^{-1})$ if $\lambda_o$ is the point at infinity).

(Condition (3) is referred to as the *normalisation condition*, and is necessary in order to prove uniqueness of the associated RHP: one says that the RHP is 'normalised at $\lambda_o$'.) Let $v(z) := (\mathrm{I} - w_{-}(z))^{-1}(\mathrm{I} + w_{+}(z))$, $z \in \widetilde{\Gamma}^{\sharp}$, be a (bounded algebraic) factorisation for $v(z)$, where $w_{\pm}(z)$ are some upper/lower, or lower/upper, triangular matrices (depending on the orientation of $\Gamma^{\sharp}$), and $w_{\pm}(z) \in \cap_{p \in \{2,\infty\}} \mathcal{L}^{p}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Gamma}^{\sharp})$ (if $\widetilde{\Gamma}^{\sharp}$ is unbounded, one requires that $w_{\pm}(z) \underset{\substack{z \to \infty \\ z \in \widetilde{\Gamma}^{\sharp}}}{=} \mathbf{0}$). Define $w(z) := w_{+}(z) + w_{-}(z)$, and introduce the (normalised at $\lambda_o$) Cauchy operators

$$\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp}) \ni f \mapsto (C^{\lambda_o}_{\pm} f)(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \Gamma^{\sharp}}} \int_{\Gamma^{\sharp}} \frac{(z' - \lambda_o) f(\zeta)}{(\zeta - \lambda_o)(\zeta - z')} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}},$$

where $\frac{(z-\lambda_o)}{(\zeta-\lambda_o)(\zeta-z)} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}$ is the Cauchy kernel normalised at $\lambda_o$ (which reduces to the 'standard' Cauchy kernel, that is, $\frac{1}{\zeta - z} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}$, in the limit $\lambda_o \to \infty$), with $C^{\lambda_o}_{\pm} \colon \mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp}) \to \mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$ bounded in operator norm[14], and $\|(C^{\lambda_o}_{\pm} f)(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})} \leqslant \text{const.} \|f(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})}$. Introduce the BC operator $C^{\lambda_o}_{w}$:

$$\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp}) \ni f \mapsto C^{\lambda_o}_{w} f := C^{\lambda_o}_{+}(f w_{-}) + C^{\lambda_o}_{-}(f w_{+}),$$

which, for $w_{\pm} \in \mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$, is bounded from $\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp}) \to \mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$, that is, $\|C^{\lambda_o}_{w}\|_{\mathcal{N}_2(\Gamma^{\sharp})} < \infty$; moreover, since $\mathbb{C} \setminus \Gamma^{\sharp}$ can be coloured by the two colours $\pm$, $C^{\lambda_o}_{\pm}$ are complementary projections [2,75,89,90], that is, $(C^{\lambda_o}_{+})^2 = C^{\lambda_o}_{+}$, $(C^{\lambda_o}_{-})^2 = -C^{\lambda_o}_{-}$, $C^{\lambda_o}_{+} C^{\lambda_o}_{-} = C^{\lambda_o}_{-} C^{\lambda_o}_{+} = \underline{\mathbf{0}}$ (the null operator), and $C^{\lambda_o}_{+} - C^{\lambda_o}_{-} = \mathbf{id}$ (the identity operator). (In the case that $C^{\lambda_o}_{+}$ and $-C^{\lambda_o}_{-}$ are complementary, the contour $\Gamma^{\sharp}$ can always be oriented in such a way that the $\pm$ regions lie on the $\pm$ sides of the contour, respectively.) The solution of the above (normalised at $\lambda_o$) RHP is given by the following integral representation.

**Lemma 5.1 (Beals and Coifman** [74]**).** *Set*

$$\mu_{\lambda_o}(z) = \mathcal{Y}_{+}(z)(\mathrm{I} + w_{+}(z))^{-1} = \mathcal{Y}_{-}(z)(\mathrm{I} - w_{-}(z))^{-1}, \quad z \in \Gamma^{\sharp}.$$

*If* $\mu_{\lambda_o} \in \mathrm{I} + \mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$ *solves the linear singular integral equation*

$$(\mathbf{id} - C^{\lambda_o}_{w})(\mu_{\lambda_o}(z) - \mathrm{I}) = C^{\lambda_o}_{w}\mathrm{I} = C^{\lambda_o}_{+}(w_{-}(z)) + C^{\lambda_o}_{-}(w_{+}(z)), \quad z \in \Gamma^{\sharp},$$

*where* $\mathbf{id}$ *is the identity operator on* $\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$*, then the solution of the* RHP $(\mathcal{Y}(z), v(z), \Gamma^{\sharp})$ *is given by*

$$\mathcal{Y}(z) = \mathrm{I} + \int_{\Gamma^{\sharp}} \frac{(z - \lambda_o)\mu_{\lambda_o}(\zeta) w(\zeta)}{(\zeta - \lambda_o)(\zeta - z)} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}, \quad z \in \mathbb{C} \setminus \Gamma^{\sharp},$$

[14] $\|C^{\lambda_o}_{\pm}\|_{\mathcal{N}_2(\Gamma^{\sharp})} < \infty$.

*where* $\mu_{\lambda_o}(z) := ((\mathbf{id} - C_w^{\lambda_o})^{-1}\mathrm{I})(z)$[15].

Recall that $\mathcal{R}^o \colon \mathbb{C} \setminus \Sigma_p^o \to \mathrm{SL}_2(\mathbb{C})$, which solves the RHP $(\mathcal{R}^o(z), v_{\mathcal{R}}^o(z), \Sigma_p^o)$ formulated in Lemma 4.8, is normalised at zero, that is, $\mathcal{R}^o(0) = \mathrm{I}$. Removing from the specification of the RHP $(\mathcal{R}^o(z), v_{\mathcal{R}}^o(z), \Sigma_p^o)$ the oriented skeletons on which the jump matrix, $v_{\mathcal{R}}^o(z)$, is equal to I, in particular (cf. Lemma 4.8), the oriented skeleton $\Sigma_p^o \setminus \cup_{l=1}^5 \Sigma_p^{o,l}$, and setting $\Sigma_p^o \setminus (\Sigma_p^o \setminus \cup_{l=1}^5 \Sigma_p^{o,l}) =: \widetilde{\Sigma}_p^o$ (see Figure 10), one arrives

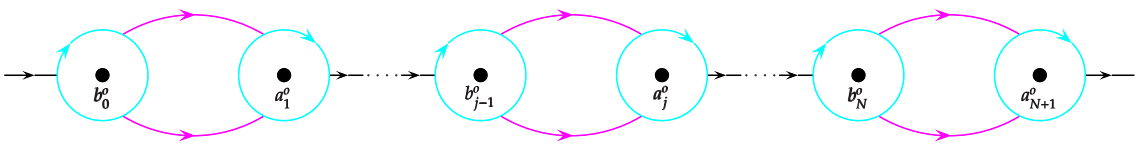


Figure 10: Oriented skeleton $\widetilde{\Sigma}_p^o := \Sigma_p^o \setminus (\Sigma_p^o \setminus \cup_{l=1}^5 \Sigma_p^{o,l})$

at the equivalent RHP $(\mathcal{R}^o(z), v_{\mathcal{R}}^o(z), \widetilde{\Sigma}_p^o)$ for $\mathcal{R}^o \colon \mathbb{C} \setminus \widetilde{\Sigma}_p^o \to \mathrm{SL}_2(\mathbb{C})$ (the normalisation at zero, of course, remains unchanged). Via the BC [74] construction discussed above, write, for $v_{\mathcal{R}}^o \colon \widetilde{\Sigma}_p^o \to \mathrm{SL}_2(\mathbb{C})$, the (bounded algebraic) factorisation

$$v_{\mathcal{R}}^o(z) := \left(\mathrm{I} - w_-^{\Sigma_{\mathcal{R}}^o}(z)\right)^{-1}\left(\mathrm{I} + w_+^{\Sigma_{\mathcal{R}}^o}(z)\right), \quad z \in \widetilde{\Sigma}_p^o :$$

taking the (so-called) trivial factorisation [76] (see pp. 293 and 294, *Proof of Theorem* 3.14 and *Proposition* 1.9; see, also, [90, 91]) $w_-^{\Sigma_{\mathcal{R}}^o}(z) \equiv \mathbf{0}$, whence $v_{\mathcal{R}}^o(z) = \mathrm{I} + w_+^{\Sigma_{\mathcal{R}}^o}(z)$, $z \in \widetilde{\Sigma}_p^o$, it follows from Lemma 5.1 that, upon normalising the Cauchy (integral) operator(s) at zero (take the limit $\lambda_o \to 0$ in Lemma 5.1), the ($\mathrm{SL}_2(\mathbb{C})$-valued) integral representation for the—unique—solution of the equivalent RHP $(\mathcal{R}^o(z), v_{\mathcal{R}}^o(z), \widetilde{\Sigma}_p^o)$ is

$$\mathcal{R}^o(z) = \mathrm{I} + \int_{\widetilde{\Sigma}_p^o} \frac{z\mu^{\Sigma_{\mathcal{R}}^o}(s) w_+^{\Sigma_{\mathcal{R}}^o}(s)}{s(s-z)} \frac{\mathrm{d}s}{2\pi \mathrm{i}}, \quad z \in \mathbb{C} \setminus \widetilde{\Sigma}_p^o, \tag{5.1}$$

where $\mu^{\Sigma_{\mathcal{R}}^o}(\cdot) \in \mathrm{I} + \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)$ solves the (linear) singular integral equation

$$(\mathbf{id} - C^0_{w^{\Sigma_{\mathcal{R}}^o}})\mu^{\Sigma_{\mathcal{R}}^o}(z) = \mathrm{I}, \quad z \in \widetilde{\Sigma}_p^o,$$

with

$$\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o) \ni f \mapsto C^0_{w^{\Sigma_{\mathcal{R}}^o}} f := C^0_-(f w_+^{\Sigma_{\mathcal{R}}^o}),$$

and

$$\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o) \ni f \mapsto (C^0_{\pm} f)(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \widetilde{\Sigma}_p^o}} \int_{\widetilde{\Sigma}_p^o} \frac{z' f(s)}{s(s-z')} \frac{\mathrm{d}s}{2\pi \mathrm{i}};$$

furthermore, $\|C^0_{\pm}\|_{\mathcal{N}_2(\widetilde{\Sigma}_p^o)} < \infty$.

**Proposition 5.2.** *Let* $\mathcal{R}^o \colon \mathbb{C} \setminus \widetilde{\Sigma}_p^o \to \mathrm{SL}_2(\mathbb{C})$ *solve the following, equivalent* RHP: (i) $\mathcal{R}^o(z)$ *is holomorphic for* $z \in \mathbb{C} \setminus \widetilde{\Sigma}_p^o$; (ii) $\mathcal{R}^o_{\pm}(z) := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \widetilde{\Sigma}_p^o}} \mathcal{R}^o(z')$ *satisfy the boundary condition*

$$\mathcal{R}^o_+(z) = \mathcal{R}^o_-(z) v_{\mathcal{R}}^o(z), \quad z \in \widetilde{\Sigma}_p^o,$$

[15]The linear singular integral equation for $\mu_{\lambda_o}(\cdot)$ stated in this Lemma 5.1 is well defined in $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$ provided that $w_{\pm}(\cdot) \in \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp}) \cap \mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$; furthermore, it is assumed that the associated RHP $(\mathcal{Y}(z), v(z), \Gamma^{\sharp})$ is solvable, that is, $\dim\ker(\mathbf{id} - C_w^{\lambda_o}) = \dim\left\{\phi \in \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp});\ (\mathbf{id} - C_w^{\lambda_o})\phi = \underline{\mathbf{0}}\right\} = \dim \varnothing = 0$ ($\Rightarrow (\mathbf{id} - C_w^{\lambda_o})^{-1}\restriction_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})}$ exists).

*where* $v^{o}_{\mathcal{R}}(z)$*, for* $z \in \widetilde{\Sigma}^{o}_{p}$*, is defined in Lemma* 4.8 *and satisfies the asymptotic (as* $n \to \infty$*) estimates given in Proposition* 5.1; (iii) $\mathcal{R}^{o}(z) \underset{z \in \mathbb{C}\setminus\widetilde{\Sigma}^{o}_{p}}{\underset{z\to 0}{=}} \mathrm{I} + \mathcal{O}(z)$; *and* (iv) $\mathcal{R}^{o}(z) \underset{z \in \mathbb{C}\setminus\widetilde{\Sigma}^{o}_{p}}{\underset{z\to\infty}{=}} \mathcal{O}(1)$. *Then:*

(1) *for* $z \in (-\infty, b^{o}_{0} - \delta^{o}_{b_0}) \cup (a^{o}_{N+1} + \delta^{o}_{a_{N+1}}, +\infty) =: \Sigma^{o,1}_{p}$,

$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{q}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,1}_{p})} \underset{n\to\infty}{=} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^{1/q}}\right), \quad q=1,2,$$
$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,1}_{p})} \underset{n\to\infty}{=} \mathcal{O}\left(f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\right),$$

*where* $c > 0$ *and* $f(n) =_{n\to\infty} \mathcal{O}(1)$;

(2) *for* $z \in (a^{o}_{j} + \delta^{o}_{a_j}, b^{o}_{j} - \delta^{o}_{b_j}) =: \Sigma^{o,2}_{p,j} \subset \cup_{l=1}^{N} \Sigma^{o,2}_{p,l} =: \Sigma^{o,2}_{p}$, $j = 1, \dots, N$,

$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{q}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,2}_{p,j})} \underset{n\to\infty}{=} \mathcal{O}\left(\frac{f_j(n)\mathrm{e}^{-(n+\frac{1}{2})c_j}}{(n+\frac{1}{2})^{1/q}}\right), \quad q=1,2,$$
$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,2}_{p,j})} \underset{n\to\infty}{=} \mathcal{O}\left(f_j(n)\mathrm{e}^{-(n+\frac{1}{2})c_j}\right),$$

*where* $c_j > 0$ *and* $f_j(n) =_{n\to\infty} \mathcal{O}(1)$;

(3) *for* $z \in \cup_{j=1}^{N+1} (J^{o,\frown}_{j} \setminus (\mathbb{C}_{+} \cap (\mathbb{U}^{o}_{\delta_{b_{j-1}}} \cup \mathbb{U}^{o}_{\delta_{a_j}}))) =: \Sigma^{o,3}_{p}$,

$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{q}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,3}_{p})} \underset{n\to\infty}{=} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^{1/q}}\right), \quad q=1,2,$$
$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,3}_{p})} \underset{n\to\infty}{=} \mathcal{O}\left(f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\right),$$

*where* $c > 0$ *and* $f(n) =_{n\to\infty} \mathcal{O}(1)$;

(4) *for* $z \in \cup_{j=1}^{N+1} (J^{o,\smile}_{j} \setminus (\mathbb{C}_{-} \cap (\mathbb{U}^{o}_{\delta_{b_{j-1}}} \cup \mathbb{U}^{o}_{\delta_{a_j}}))) =: \Sigma^{o,4}_{p}$,

$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{q}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,4}_{p})} \underset{n\to\infty}{=} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^{1/q}}\right), \quad q=1,2,$$
$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,4}_{p})} \underset{n\to\infty}{=} \mathcal{O}\left(f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\right),$$

*where* $c > 0$ *and* $f(n) =_{n\to\infty} \mathcal{O}(1)$; *and*

(5) *for* $z \in \cup_{j=1}^{N+1} (\partial\mathbb{U}^{o}_{\delta_{b_{j-1}}} \cup \partial\mathbb{U}^{o}_{\delta_{a_j}}) =: \Sigma^{o,5}_{p}$,

$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{q}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,5}_{p})} \underset{n\to\infty}{=} \mathcal{O}\left((n+\tfrac{1}{2})^{-1} f(n)\right), \quad q \in \{1,2,\infty\},$$

*where* $f(n) =_{n\to\infty} \mathcal{O}(1)$.

*Furthermore,*

$$\|C^{0}_{w^{\Sigma^{o}_{\mathcal{R}}}}\|_{\mathcal{N}_r(\widetilde{\Sigma}^{o}_{p})} \underset{n\to\infty}{\leqslant} \mathcal{O}\left((n+\tfrac{1}{2})^{-1+\frac{1}{r}} f(n)\right), \quad r \in \{2,\infty\},$$

*where* $f(n) =_{n\to\infty} \mathcal{O}(1)$*; in particular,* $(\mathbf{id} - C^{0}_{w^{\Sigma^{o}_{\mathcal{R}}}})^{-1} \restriction_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}$ *exists, that is,*

$$\|(\mathbf{id} - C^{0}_{w^{\Sigma^{o}_{\mathcal{R}}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})} \underset{n\to\infty}{=} \mathcal{O}(1),$$

*and it can be expanded in a Neumann series.*

*Proof.* Without loss of generality, assume that $0 \in \Sigma^{o,1}_{p}$ (cf. Proposition 5.1). Recall that $w^{\Sigma^{o}_{\mathcal{R}}}_{+}(z) = v^{o}_{\mathcal{R}}(z) - \mathrm{I}$, $z \in \widetilde{\Sigma}^{o}_{p}$. For $z \in \Sigma^{o,1}_{p}$, using the asymptotic (as $n \to \infty$) estimate for $v^{o}_{\mathcal{R}}(z)$ given in item (1) of Proposition 5.1, one gets that

$$\|w^{\Sigma^{o}_{\mathcal{R}}}_{+}(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o,1}_{p})} := \max_{i,j=1,2} \sup_{z\in\Sigma^{o,1}_{p}} |(w^{\Sigma^{o}_{\mathcal{R}}}_{+}(z))_{ij}| \underset{n\to\infty}{=} \mathcal{O}\left(f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\right),$$

$$\begin{aligned}
\|w_{+}^{\Sigma_{\mathcal{R}}^{o}}(\cdot)\|_{\mathcal{L}^{1}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p}^{o,1})} &:= \int_{\Sigma_{p}^{o,1}} |w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z)|\,|\mathrm{d}z| = \int_{(\Sigma_{p}^{o,1}\setminus\mathbb{U}_0^o)\cup\mathbb{U}_0^o} |w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z)|\,|\mathrm{d}z| \\
&= \left(\int_{\Sigma_{p}^{o,1}\setminus\mathbb{U}_0^o} + \int_{\mathbb{U}_0^o}\right)\left(\sum_{i,j=1}^{2}(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}\overline{(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}}\right)^{1/2} |\mathrm{d}z| \\
&\underset{n\to\infty}{=} \mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big) + \mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big) \\
&\underset{n\to\infty}{=} \mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big)
\end{aligned}$$

($|\mathrm{d}z|$ denotes arc length), and

$$\begin{aligned}
\|w_{+}^{\Sigma_{\mathcal{R}}^{o}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p}^{o,1})} &:= \left(\int_{\Sigma_{p}^{o,1}} |w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z)|^2\,|\mathrm{d}z|\right)^{1/2} = \left(\int_{(\Sigma_{p}^{o,1}\setminus\mathbb{U}_0^o)\cup\mathbb{U}_0^o} |w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z)|^2\,|\mathrm{d}z|\right)^{1/2} \\
&= \left(\left(\int_{\Sigma_{p}^{o,1}\setminus\mathbb{U}_0^o} + \int_{\mathbb{U}_0^o}\right)\left(\sum_{i,j=1}^{2}(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}\overline{(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}}\right) |\mathrm{d}z|\right)^{1/2} \\
&\underset{n\to\infty}{=} \Big(\mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big) + \mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big)\Big)^{1/2} \\
&\underset{n\to\infty}{=} \mathcal{O}\Big((n+\tfrac{1}{2})^{-1/2}f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big),
\end{aligned}$$

where $c>0$ and $f(n)=_{n\to\infty}\mathcal{O}(1)$.

For $z\in(a_j^o+\delta_{a_j}^o, b_j^o-\delta_{b_j}^o)=:\Sigma_{p,j}^{o,2}\subset\cup_{l=1}^{N}\Sigma_{p,l}^{o,2}=:\Sigma_p^{o,2}$, $j=1,\dots,N$, using the asymptotic (as $n\to\infty$) estimate for $v_{\mathcal{R}}^o(z)$ given in item (2) of Proposition 5.1, one gets that

$$\|w_{+}^{\Sigma_{\mathcal{R}}^{o}}(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p,j}^{o,2})} := \max_{k,m=1,2}\sup_{z\in\Sigma_{p,j}^{o,2}}|(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{km}| \underset{n\to\infty}{=} \mathcal{O}\Big(f_j(n)\mathrm{e}^{-(n+\frac{1}{2})c_j}\Big),$$

$$\begin{aligned}
\|w_{+}^{\Sigma_{\mathcal{R}}^{o}}(\cdot)\|_{\mathcal{L}^{1}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p,j}^{o,2})} &:= \int_{\Sigma_{p,j}^{o,2}} |w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z)|\,|\mathrm{d}z| = \int_{\Sigma_{p,j}^{o,2}}\left(\sum_{i,j=1}^{2}(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}\overline{(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}}\right)^{1/2} |\mathrm{d}z| \\
&\underset{n\to\infty}{=} \mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f_j(n)\mathrm{e}^{-(n+\frac{1}{2})c_j}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\|w_{+}^{\Sigma_{\mathcal{R}}^{o}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p,j}^{o,2})} &:= \left(\int_{\Sigma_{p,j}^{o,2}} |w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z)|^2\,|\mathrm{d}z|\right)^{1/2} = \left(\int_{\Sigma_{p,j}^{o,2}}\sum_{k,l=1}^{2}(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{kl}\overline{(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{kl}}\,|\mathrm{d}z|\right)^{1/2} \\
&\underset{n\to\infty}{=} \mathcal{O}\Big((n+\tfrac{1}{2})^{-1/2}f_j(n)\mathrm{e}^{-(n+\frac{1}{2})c_j}\Big),
\end{aligned}$$

where $c_j>0$ and $f_j(n)=_{n\to\infty}\mathcal{O}(1)$, $j=1,\dots,N$.

For $z\in\cup_{j=1}^{N+1}(J_j^{o,\frown}\setminus(\mathbb{C}_+\cap(\mathbb{U}_{\delta_{b_{j-1}}}^o\cup\mathbb{U}_{\delta_{a_j}}^o)))=:\Sigma_p^{o,3}$ $(\subset\widetilde{\Sigma}_p^o)$, using the asymptotic (as $n\to\infty$) estimate for $v_{\mathcal{R}}^o(z)$ given in item (3) of Proposition 5.1, one gets that

$$\|w_{+}^{\Sigma_{\mathcal{R}}^{o}}(\cdot)\|_{\mathcal{L}^{\infty}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p}^{o,3})} := \max_{i,j=1,2}\sup_{z\in\Sigma_{p}^{o,3}}|(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}| \underset{n\to\infty}{=} \mathcal{O}\Big(f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big),$$

$$\begin{aligned}
\|w_{+}^{\Sigma_{\mathcal{R}}^{o}}(\cdot)\|_{\mathcal{L}^{1}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p}^{o,3})} &:= \int_{\Sigma_{p}^{o,3}} |w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z)|\,|\mathrm{d}z| = \int_{\Sigma_{p}^{o,3}}\left(\sum_{i,j=1}^{2}(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}\overline{(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}}\right)^{1/2} |\mathrm{d}z| \\
&\underset{n\to\infty}{=} \mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big),
\end{aligned}$$

and

$$\|w_{+}^{\Sigma_{\mathcal{R}}^{o}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{p}^{o,3})} := \left(\int_{\Sigma_{p}^{o,3}} |w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z)|^2\,|\mathrm{d}z|\right)^{1/2} = \left(\int_{\Sigma_{p}^{o,3}}\sum_{i,j=1}^{2}(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}\overline{(w_{+}^{\Sigma_{\mathcal{R}}^{o}}(z))_{ij}}\,|\mathrm{d}z|\right)^{1/2}$$

$$\underset{n\to\infty}{=}\mathcal{O}\Big((n+\tfrac{1}{2})^{-1/2}f(n)\mathrm{e}^{-(n+\frac{1}{2})c}\Big),$$

where $c>0$ and $f(n)=_{n\to\infty}\mathcal{O}(1)$: the above analysis applies, *mutatis mutandis*, for the analogous estimates on $\Sigma_p^{o,4}:=\cup_{j=1}^{N+1}(J_j^{o,\smile}\setminus(\mathbb{C}_-\cap(\mathbb{U}^o_{\delta_{b_{j-1}}}\cup\mathbb{U}^o_{\delta_{a_j}})))\ni z$.

For $z\in\cup_{j=1}^{N+1}(\partial\mathbb{U}^o_{\delta_{b_{j-1}}}\cup\partial\mathbb{U}^o_{\delta_{a_j}})=:\Sigma_p^{o,5}\ (\subset\widetilde{\Sigma}_p^o)$, using the $(2(N+1))$ asymptotic (as $n\to\infty$) estimates for $v^o_{\mathcal{R}}(z)$ given in item (5) of Proposition 5.1, one gets that

$$\|w_+^{\Sigma^o_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{o,5})}:=\max_{i,j=1,2}\sup_{z\in\Sigma_p^{o,5}}|(w_+^{\Sigma^o_{\mathcal{R}}}(z))_{ij}|\underset{n\to\infty}{=}\mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\Big),$$

$$\begin{aligned}\|w_+^{\Sigma^o_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^1_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{o,5})}&:=\int_{\Sigma_p^{o,5}}|w_+^{\Sigma^o_{\mathcal{R}}}(z)|\,|\mathrm{d}z|=\int_{\cup_{j=1}^{N+1}(\partial\mathbb{U}^o_{\delta_{b_{j-1}}}\cup\partial\mathbb{U}^o_{\delta_{a_j}})}|w_+^{\Sigma^o_{\mathcal{R}}}(z)|\,|\mathrm{d}z|\\&=\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^o_{\delta_{b_{k-1}}}}+\int_{\partial\mathbb{U}^o_{\delta_{a_k}}}\right)\left(\sum_{i,j=1}^{2}(w_+^{\Sigma^o_{\mathcal{R}}}(z))_{ij}\overline{(w_+^{\Sigma^o_{\mathcal{R}}}(z))_{ij}}\right)^{1/2}|\mathrm{d}z|,\end{aligned}$$

whence, (cf. Lemma 4.5) using the fact that the respective factors $(\gamma^o(0))^{-1}\gamma^o(z)\pm\gamma^o(0)(\gamma^o(z))^{-1}$ and $\boldsymbol{\theta}^o(\pm\boldsymbol{u}^o(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o\pm\boldsymbol{d}_o)$ are uniformly bounded (with respect to $z$) in compact intervals outside open intervals surrounding the end-points of the support of the 'odd' equilibrium measure, one arrives at

$$\begin{aligned}\|w_+^{\Sigma^o_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^1_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{o,5})}&\underset{n\to\infty}{=}\frac{1}{(n+\frac{1}{2})}\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^o_{\delta_{b_{k-1}}}}\frac{|\star^o_{b_{k-1}}(z;n)|}{(z-b^o_{k-1})^{3/2}}|\mathrm{d}z|+\int_{\partial\mathbb{U}^o_{\delta_{a_k}}}\frac{|\star^o_{a_k}(z;n)|}{(z-a^o_k)^{3/2}}|\mathrm{d}z|\right)\\&\underset{n\to\infty}{=}\mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\Big),\end{aligned}$$

and, similarly,

$$\begin{aligned}\|w_+^{\Sigma^o_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma_p^{o,5})}&:=\left(\int_{\Sigma_p^{o,5}}|w_+^{\Sigma^o_{\mathcal{R}}}(z)|^2\,|\mathrm{d}z|\right)^{1/2}=\left(\int_{\cup_{j=1}^{N+1}(\partial\mathbb{U}^o_{\delta_{b_{j-1}}}\cup\partial\mathbb{U}^o_{\delta_{a_j}})}|w_+^{\Sigma^o_{\mathcal{R}}}(z)|^2\,|\mathrm{d}z|\right)^{1/2}\\&=\left(\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^o_{\delta_{b_{k-1}}}}+\int_{\partial\mathbb{U}^o_{\delta_{a_k}}}\right)\sum_{i,j=1}^{2}(w_+^{\Sigma^o_{\mathcal{R}}}(z))_{ij}\overline{(w_+^{\Sigma^o_{\mathcal{R}}}(z))_{ij}}\,|\mathrm{d}z|\right)^{1/2}\\&\underset{n\to\infty}{=}\left(\frac{1}{(n+\frac{1}{2})^2}\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^o_{\delta_{b_{k-1}}}}\frac{|\star^o_{b_{k-1}}(z;n)|}{(z-b^o_{k-1})^{3}}|\mathrm{d}z|+\int_{\partial\mathbb{U}^o_{\delta_{a_k}}}\frac{|\star^o_{a_k}(z;n)|}{(z-a^o_k)^{3}}|\mathrm{d}z|\right)\right)^{1/2}\\&\underset{n\to\infty}{=}\mathcal{O}\Big((n+\tfrac{1}{2})^{-1}f(n)\Big),\end{aligned}$$

where $f(n)=_{n\to\infty}\mathcal{O}(1)$.

Recall that $C^0_{w^{\Sigma^o_{\mathcal{R}}}}f:=C^0_-(fw_+^{\Sigma^o_{\mathcal{R}}})$, where $(C^0_-g)(z):=\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}^o_p}}\int_{\widetilde{\Sigma}^o_p}\frac{z'g(s)}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$, with $-\widetilde{\Sigma}^o_p$ shorthand for 'the $-$ side of $\widetilde{\Sigma}^o_p$'. For the $\|C^0_{w^{\Sigma^o_{\mathcal{R}}}}\|_{\mathcal{N}_\infty(\widetilde{\Sigma}^o_p)}$ norm, one proceeds as follows:

$$\begin{aligned}\|C^0_{w^{\Sigma^o_{\mathcal{R}}}}g\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}&:=\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}^o_p}|(C^0_{w^{\Sigma^o_{\mathcal{R}}}}g)_{jl}(z)|=\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}^o_p}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}^o_p}}\int_{\widetilde{\Sigma}^o_p}\frac{z'(g(s)w_+^{\Sigma^o_{\mathcal{R}}}(s))_{jl}}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|\\&\leqslant\|g(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}^o_p}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}^o_p}}\int_{\widetilde{\Sigma}^o_p}\frac{z'(w_+^{\Sigma^o_{\mathcal{R}}}(s))_{jl}}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|\\&\leqslant\|g(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}^o_p}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}^o_p}}\left(\int_{\Sigma_p^{o,1}\setminus\mathbb{U}^o_0}+\int_{\mathbb{U}^o_0}+\sum_{k=1}^{N+1}\int_{\Sigma^{o,2}_{p,k}}+\int_{\Sigma_p^{o,3}}\right.\right.\end{aligned}$$

$$+\left.\int_{\Sigma_p^{o,4}}+\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^o_{\delta_{b_{k-1}}}}+\int_{\partial\mathbb{U}^o_{a_k}}\right)\right)\frac{z'(w_+^{\Sigma^o_{\mathcal{R}}}(s))_{jl}}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|$$

$$\underset{n\to\infty}{\leqslant}\|g(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\max_{j,l=1,2}\sup_{z\in\widetilde{\Sigma}^o_p}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}^o_p}}\left(\int_{\Sigma_p^{o,1}\setminus\mathbb{U}^o_0}\frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})c_\infty|s|}f_\infty(n)))_{jl}z'}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right.\right.$$

$$+\int_{\mathbb{U}^o_0}\frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})c_0|s|^{-1}}f_0(n)))_{jl}z'}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\sum_{k=1}^{N}\int_{\Sigma_{p,k}^{o,2}}\frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})c_k(s-a_k^o)}f_k(n)))_{jl}z'}{s(s-z')}$$

$$\times\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\int_{\Sigma_p^{o,3}}\frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})\hat{c}|s|}\hat{f}(n)))_{jl}z'}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\int_{\Sigma_p^{o,4}}\frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})\breve{c}|s|}\breve{f}(n)))_{jl}z'}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$$

$$+\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^o_{\delta_{b_{k-1}}}}\left(\mathcal{O}\left(\frac{z'\overset{o}{\mathfrak{M}}{}^\infty(s)\left(\begin{smallmatrix}*&*\\ *&*\end{smallmatrix}\right)(\overset{o}{\mathfrak{M}}{}^\infty(s))^{-1}}{(n+\frac{1}{2})s(s-z')(s-b^o_{k-1})^{3/2}G^o_{b_{k-1}}(s)}\right)\right)_{jl}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right.$$

$$\left.\left.\left.+\int_{\partial\mathbb{U}^o_{\delta_{a_k}}}\left(\mathcal{O}\left(\frac{z'\overset{o}{\mathfrak{M}}{}^\infty(s)\left(\begin{smallmatrix}*&*\\ *&*\end{smallmatrix}\right)(\overset{o}{\mathfrak{M}}{}^\infty(s))^{-1}}{(n+\frac{1}{2})s(s-z')(s-a^o_k)^{3/2}G^o_{a_k}(s)}\right)\right)_{jl}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right)\right)\right|,$$

whence, taking note of the partial fraction decomposition $\frac{z'}{s(s-z')}=-\frac{1}{s}+\frac{1}{s-z'}$, and (cf. Lemma 4.5) using the fact that the respective factors $(\gamma^o(0))^{-1}\gamma^o(z)\pm\gamma^o(0)(\gamma^o(z))^{-1}$ and $\boldsymbol{\theta}^o(\pm\boldsymbol{u}^o(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o\pm\boldsymbol{d}_o)$ are uniformly bounded (with respect to $z$) in compact intervals outside open intervals surrounding the end-points of the support of the 'odd' equilibrium measure, one arrives at, after a straightforward integration argument and an application of the Maximum Length (ML) Theorem,

$$\|C^0_{w^{\Sigma^o_{\mathcal{R}}}}g\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\underset{n\to\infty}{\leqslant}\|g(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\left(\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})\min\{1,\operatorname{dist}(z,\widetilde{\Sigma}^o_p)\}}\right)\right.$$

$$\left.+\mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})\min\{1,\operatorname{dist}(z,\widetilde{\Sigma}^o_p)\}}\right)\right)\underset{n\to\infty}{=}\|g(\cdot)\|_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right),$$

where $\operatorname{dist}(z,\widetilde{\Sigma}^o_p):=\inf\left\{|z-r|;\,r\in\widetilde{\Sigma}^o_p,\,z\in\mathbb{C}\setminus\widetilde{\Sigma}^o_p\right\}$ $(>0)$, and $f(n)=_{n\to\infty}\mathcal{O}(1)$, whence one obtains the asymptotic (as $n\to\infty$) estimate for $\|C^0_{w^{\Sigma^o_{\mathcal{R}}}}\|_{\mathcal{N}_\infty(\widetilde{\Sigma}^o_p)}$ stated in the Proposition. Similarly, for $\|C^0_{w^{\Sigma^o_{\mathcal{R}}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}$, as $n\to\infty$:

$$\|C^0_{w^{\Sigma^o_{\mathcal{R}}}}g\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}:=\left(\int_{\widetilde{\Sigma}^o_p}|(C^0_{w^{\Sigma^o_{\mathcal{R}}}}g)(z)|^2\,|\mathrm{d}z|\right)^{1/2}=\left(\int_{\widetilde{\Sigma}^o_p}\sum_{j,l=1}^{2}(C^0_{w^{\Sigma^o_{\mathcal{R}}}}g)_{jl}(z)\overline{(C^0_{w^{\Sigma^o_{\mathcal{R}}}}g)_{jl}(z)}\,|\mathrm{d}z|\right)^{1/2}$$

$$=\left(\int_{\widetilde{\Sigma}^o_p}\sum_{j,l=1}^{2}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}^o_p}}\int_{\widetilde{\Sigma}^o_p}\frac{z'(g(s)w_+^{\Sigma^o_{\mathcal{R}}}(s))_{jl}}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|^2|\mathrm{d}z|\right)^{1/2}$$

$$\leqslant\|g(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\left(\int_{\widetilde{\Sigma}^o_p}\sum_{j,l=1}^{2}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}^o_p}}\int_{\widetilde{\Sigma}^o_p}\frac{z'(w_+^{\Sigma^e_{\mathcal{R}}}(s))_{jl}}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|^2|\mathrm{d}z|\right)^{1/2}$$

$$\leqslant\|g(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\left(\int_{\widetilde{\Sigma}^o_p}\sum_{j,l=1}^{2}\left|\lim_{\substack{z'\to z\\ z'\in-\widetilde{\Sigma}^o_p}}\left(\int_{\Sigma_p^{o,1}\setminus\mathbb{U}^o_0}+\int_{\mathbb{U}^o_0}+\sum_{k=1}^{N+1}\int_{\Sigma_{p,k}^{o,2}}+\int_{\Sigma_p^{o,3}}\right.\right.\right.$$

$$\left.\left.\left.+\int_{\Sigma_p^{o,4}}+\sum_{k=1}^{N+1}\left(\int_{\partial\mathbb{U}^o_{\delta_{b_{k-1}}}}+\int_{\partial\mathbb{U}^o_{a_k}}\right)\right)\frac{z'(w_+^{\Sigma^o_{\mathcal{R}}}(s))_{jl}}{s(s-z')}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right|^2|\mathrm{d}z|\right)^{1/2}$$

$$\begin{aligned}
&\underset{n\to\infty}{\leqslant} \|g(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)} \left( \int_{\widetilde{\Sigma}_p^o} \sum_{j,l=1}^{2} \left| \lim_{\substack{z'\to z\\ z'\in -\widetilde{\Sigma}_p^o}} \left( \int_{\Sigma_p^{o,1}\setminus \mathbb{U}_0^o} \frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})c_\infty|s|} f_\infty(n)))_{jl} z'}{s(s-z')} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \right.\right.\right. \\
&+ \int_{\mathbb{U}_0^o} \frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})c_0|s|^{-1}} f_0(n)))_{jl} z'}{s(s-z')} \frac{\mathrm{d}s}{2\pi\mathrm{i}} + \sum_{k=1}^{N} \int_{\Sigma_{p,k}^{o,2}} \frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})c_k(s-a_k^o)} f_k(n)))_{jl} z'}{s(s-z')} \\
&\times \frac{\mathrm{d}s}{2\pi\mathrm{i}} + \int_{\Sigma_p^{o,3}} \frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})\widehat{c}|s|} \widehat{f}(n)))_{jl} z'}{s(s-z')} \frac{\mathrm{d}s}{2\pi\mathrm{i}} + \int_{\Sigma_p^{o,4}} \frac{(\mathcal{O}(\mathrm{e}^{-(n+\frac{1}{2})\check{c}|s|} \check{f}(n)))_{jl} z'}{s(s-z')} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \\
&+ \sum_{k=1}^{N+1} \left( \int_{\partial \mathbb{U}^o_{\delta_{b_{k-1}}}} \left( \mathcal{O}\left( \frac{z' \overset{o}{\mathfrak{M}}{}^{\infty}(s) \left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right) (\overset{o}{\mathfrak{M}}{}^{\infty}(s))^{-1}}{(n+\frac{1}{2}) s(s-z')(s-b_{k-1}^o)^{3/2} G^o_{b_{k-1}}(s)} \right) \right)_{jl} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \right. \\
&\left.\left.\left.\left. + \int_{\partial \mathbb{U}^o_{\delta_{a_k}}} \left( \mathcal{O}\left( \frac{z' \overset{o}{\mathfrak{M}}{}^{\infty}(s) \left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right) (\overset{o}{\mathfrak{M}}{}^{\infty}(s))^{-1}}{(n+\frac{1}{2}) s(s-z')(s-a_k^o)^{3/2} G^o_{a_k}(s)} \right) \right)_{jl} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \right) \right) \right|^2 |\mathrm{d}z| \right)^{1/2},
\end{aligned}$$

whence, taking note of the partial fraction decomposition $\frac{z'}{s(s-z')} = -\frac{1}{s} + \frac{1}{s-z'}$, and (cf. Lemma 4.5) using the fact that the respective factors $(\gamma^o(0))^{-1}\gamma^o(z) \pm \gamma^o(0)(\gamma^o(z))^{-1}$ and $\boldsymbol{\theta}^o(\pm \boldsymbol{u}^o(z) - \frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o \pm \boldsymbol{d}_o)$ are uniformly bounded (with respect to $z$) in compact intervals outside open intervals surrounding the end-points of the support of the 'odd' equilibrium measure, one arrives at, after a straightforward integration argument and an application of the ML Theorem,

$$\begin{aligned}
\|C^0_{w^{\Sigma^o_{\mathcal{R}}}} g\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)} &\underset{n\to\infty}{\leqslant} \|g(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)} \left( \mathcal{O}\left( \frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^{1/2} \min\{1, \mathrm{dist}(z, \widetilde{\Sigma}_p^o)\}} \right) \right. \\
&\left. + \mathcal{O}\left( \frac{f(n)}{(n+\frac{1}{2})^{1/2} \min\{1, \mathrm{dist}(z, \widetilde{\Sigma}_p^o)\}} \right) \right) \underset{n\to\infty}{=} \|g(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)} \mathcal{O}\left( \frac{f(n)}{\sqrt{n+\frac{1}{2}}} \right),
\end{aligned}$$

where $f(n) =_{n\to\infty} \mathcal{O}(1)$, whence one obtains the asymptotic (as $n \to \infty$) inequality for $\|C^0_{w^{\Sigma^o_{\mathcal{R}}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}_p^o)}$ stated in the Proposition. The above analysis establishes the fact that, as $n\to\infty$, $C^0_{w^{\Sigma^o_{\mathcal{R}}}} \in \mathcal{N}_2(\widetilde{\Sigma}_p^o)$, with operator norm $\|C^0_{w^{\Sigma^o_{\mathcal{R}}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}_p^o)} =_{n\to\infty} \mathcal{O}((n+1/2)^{-1/2} f(n))$, where $f(n) =_{n\to\infty} \mathcal{O}(1)$; due to a well-known result for bounded linear operators in Hilbert space [92], it follows, thus, that $(\mathbf{id} - C^0_{w^{\Sigma^o_{\mathcal{R}}}})^{-1}\restriction_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)}$ exists, and $(\mathbf{id} - C^0_{w^{\Sigma^o_{\mathcal{R}}}})\restriction_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)}$ can be inverted by a Neumann series (as $n\to\infty$), with $\|(\mathbf{id} - C^0_{w^{\Sigma^o_{\mathcal{R}}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}_p^o)} \leqslant_{n\to\infty} (1 - \|C^0_{w^{\Sigma^o_{\mathcal{R}}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}_p^o)})^{-1} =_{n\to\infty} \mathcal{O}(1)$. □

**Lemma 5.2.** *Set* $\Sigma^o_{\circlearrowright} := \Sigma_p^{o,5}$ $(= \cup_{j=1}^{N+1}(\partial \mathbb{U}^o_{\delta_{b_{j-1}}} \cup \partial \mathbb{U}^o_{\delta_{a_j}}))$ *and* $\Sigma^o_{\blacksquare} := \widetilde{\Sigma}_p^o \setminus \Sigma^o_{\circlearrowright}$, *and let* $\mathcal{R}^o \colon \mathbb{C} \setminus \widetilde{\Sigma}_p^o \to \mathrm{SL}_2(\mathbb{C})$ *solve the (equivalent)* RHP $(\mathcal{R}^o(z), \upsilon^o_{\mathcal{R}}(z), \widetilde{\Sigma}_p^o)$ *formulated in Proposition* 5.2 *with integral representation given by Equation* (5.1). *Let the asymptotic (as $n\to\infty$) estimates and bounds given in Propositions* 5.1 *and* 5.2 *be valid. Then, uniformly for compact subsets of* $\mathbb{C} \setminus \widetilde{\Sigma}_p^o \ni z$,

$$\mathcal{R}^o(z) \underset{\substack{n\to\infty\\ z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^o}}{=} \mathrm{I} + \int_{\Sigma^o_{\circlearrowright}} \frac{z w_+^{\Sigma^o_{\circlearrowright}}(s)}{s(s-z)} \frac{\mathrm{d}s}{2\pi\mathrm{i}} + \mathcal{O}\left( \frac{f(n)}{(n+\frac{1}{2})^2 \min\{1, \mathrm{dist}(z, \widetilde{\Sigma}_p^o)\}} \right),$$

*where* $w_+^{\Sigma^o_{\circlearrowright}}(z) := w_+^{\Sigma^o_{\mathcal{R}}}(z)\restriction_{\Sigma^o_{\circlearrowright}}$, *and* $(f(n))_{ij} =_{n\to\infty} \mathcal{O}(1)$, $i, j = 1, 2$.

*Proof.* Define $\Sigma^o_{\circlearrowright}$ and $\Sigma^o_{\blacksquare}$ as in the Lemma, and write $\widetilde{\Sigma}_p^o = (\widetilde{\Sigma}_p^o \setminus \Sigma^o_{\circlearrowright}) \cup \Sigma^o_{\circlearrowright} := \Sigma^o_{\blacksquare} \cup \Sigma^o_{\circlearrowright}$ (with $\Sigma^o_{\blacksquare} \cap \Sigma^o_{\circlearrowright} = \varnothing$). Recall, from Equation (5.1), the integral representation for $\mathcal{R}^o \colon \mathbb{C} \setminus \widetilde{\Sigma}_p^o \to \mathrm{SL}_2(\mathbb{C})$:

$$\mathcal{R}^o(z) = \mathrm{I} + \int_{\widetilde{\Sigma}_p^o} \frac{z \mu^{\Sigma^o_{\mathcal{R}}}(s) w_+^{\Sigma^o_{\mathcal{R}}}(s)}{s(s-z)} \frac{\mathrm{d}s}{2\pi\mathrm{i}}, \quad z \in \mathbb{C} \setminus \widetilde{\Sigma}_p^o.$$

Using the linearity property of the Cauchy integral operator $C^{0}_{w^{\Sigma^{o}_{\mathcal{R}}}}$, one shows that $C^{0}_{w^{\Sigma^{o}_{\mathcal{R}}}} = C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}} + C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}$. Via a repeated application of the second resolvent identity[16]:

$$
\begin{aligned}
\mu^{\Sigma^{o}_{\mathcal{R}}}(z) &= \mathrm{I}+((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\mathcal{R}}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\mathcal{R}}}}\mathrm{I})(z)=\mathrm{I}+((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}+C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})\mathrm{I})(z)\\
&= \mathrm{I}+((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I})(z)+((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I})(z)\\
&= \mathrm{I}+(((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}))^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I})(z)\\
&+ (((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}))^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I})(z)\\
&= \mathrm{I}+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}+(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I})(z)\\
&+ ((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}+(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I})(z)\\
&= \mathrm{I}+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)\\
&= \mathrm{I}+((\mathbf{id}+(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}+(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}+(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}+(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)\\
&= \mathrm{I}+(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I})(z)+(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I})(z)+((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\\
&\times (C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)\\
&+ ((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)\\
&= \mathrm{I}+(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I})(z)+(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I})(z)+((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)+((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\\
&\times(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\\
&\times C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\\
&\times (\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}\\
&\times (\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(z)+((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\\
&\times (\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\\
&\times (C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(z);
\end{aligned}
$$

hence, recalling the integral representation for $\mathcal{R}^{o}(z)$ given above, one arrives at, for $\mathbb{C}\setminus\widetilde{\Sigma}^{o}_{p}\ni z$, upon using the partial fraction decomposition $\frac{z}{s(s-z)}=-\frac{1}{s}+\frac{1}{s-z}$,

$$
\mathcal{R}^{o}(z)-\mathrm{I}-\int_{\Sigma^{o}_{\circlearrowright}}\frac{zw^{\Sigma^{o}_{\circlearrowright}}_{+}(s)}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}=\int_{\Sigma^{o}_{\blacksquare}}w^{\Sigma^{o}_{\blacksquare}}_{+}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\sum_{k=1}^{8}I^{o}_{k}, \tag{5.2}
$$

where $w^{\Sigma^{o}_{\circlearrowright}}_{+}(z):=w^{\Sigma^{o}_{\mathcal{R}}}_{+}(z)\restriction_{\Sigma^{o}_{\circlearrowright}}$, $w^{\Sigma^{o}_{\blacksquare}}_{+}(z):=w^{\Sigma^{o}_{\mathcal{R}}}_{+}(z)\restriction_{\Sigma^{o}_{\blacksquare}}$,

$$
I^{o}_{1}:=\int_{\widetilde{\Sigma}^{o}_{p}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I})(s)w^{\Sigma^{o}_{\mathcal{R}}}_{+}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}}, \qquad I^{o}_{2}:=\int_{\widetilde{\Sigma}^{o}_{p}}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I})(s)w^{\Sigma^{o}_{\mathcal{R}}}_{+}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}},
$$

[16]For general operators $\mathcal{A}$ and $\mathcal{B}$, if $(\mathbf{id}-\mathcal{A})^{-1}$ and $(\mathbf{id}-\mathcal{B})^{-1}$ exist, then $(\mathbf{id}-\mathcal{B})^{-1}-(\mathbf{id}-\mathcal{A})^{-1}=(\mathbf{id}-\mathcal{B})^{-1}(\mathcal{B}-\mathcal{A})(\mathbf{id}-\mathcal{A})^{-1}$ [92].

$$
\begin{aligned}
I_3^o &:= \int_{\widetilde{\Sigma}_p^o} ((\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}C^0_{w^{\Sigma^o_\blacksquare}}(C^0_{w^{\Sigma^o_\blacksquare}}\mathrm{I}))(s)w_+^{\Sigma^o_{\mathcal{R}}}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}},\\
I_4^o &:= \int_{\widetilde{\Sigma}_p^o} ((\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}(C^0_{w^{\Sigma^o_\circlearrowright}}\mathrm{I}))(s)w_+^{\Sigma^o_{\mathcal{R}}}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}},\\
I_5^o &:= \int_{\widetilde{\Sigma}_p^o} ((\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}C^0_{w^{\Sigma^o_\blacksquare}}C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\\
&\quad\times C^0_{w^{\Sigma^o_\circlearrowright}}(C^0_{w^{\Sigma^o_\blacksquare}}\mathrm{I}))(s)w_+^{\Sigma^o_{\mathcal{R}}}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}},\\
I_6^o &:= \int_{\widetilde{\Sigma}_p^o} ((\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}\\
&\quad\times C^0_{w^{\Sigma^o_\blacksquare}}(C^0_{w^{\Sigma^o_\circlearrowright}}\mathrm{I}))(s)w_+^{\Sigma^o_{\mathcal{R}}}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}},\\
I_7^o &:= \int_{\widetilde{\Sigma}_p^o} ((\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}C^0_{w^{\Sigma^o_\blacksquare}}C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\\
&\quad\times C^0_{w^{\Sigma^o_\circlearrowright}}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}C^0_{w^{\Sigma^o_\blacksquare}}(C^0_{w^{\Sigma^o_\blacksquare}}\mathrm{I}))(s)w_+^{\Sigma^o_{\mathcal{R}}}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}},\\
I_8^o &:= \int_{\widetilde{\Sigma}_p^o} ((\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}\\
&\quad\times C^0_{w^{\Sigma^o_\blacksquare}}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}(C^0_{w^{\Sigma^o_\circlearrowright}}\mathrm{I}))(s)w_+^{\Sigma^o_{\mathcal{R}}}(s)\left(\frac{1}{s-z}-\frac{1}{s}\right)\frac{\mathrm{d}s}{2\pi\mathrm{i}}.
\end{aligned}
$$

One now proceeds to estimate, as $n\to\infty$, and without loss of generality, the respective terms on the right-hand side of Equation (5.2) corresponding to the (standard) Cauchy kernel, $\frac{1}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$, using the estimates and bounds given in Propositions 5.1 and 5.2.

$$
\left|\int_{\Sigma^o_\blacksquare}\frac{w_+^{\Sigma^o_\blacksquare}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\right| \leqslant \int_{\Sigma^o_\blacksquare}\frac{|w_+^{\Sigma^o_\blacksquare}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi} \leqslant \frac{\|w_+^{\Sigma^o_\blacksquare}(\cdot)\|_{\mathcal{L}^1_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\blacksquare)}}{2\pi\,\mathrm{dist}(z,\Sigma^o_\blacksquare)} \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})\,\mathrm{dist}(z,\widetilde{\Sigma}_p^o)}\right),
$$

where, here and below, ($f(n)>0$ and) $f(n)=_{n\to\infty}\mathcal{O}(1)$, and $c>0$. One estimates the 'Cauchy part' of $I_1^o$, denoted $I_1^{o,C}$, as follows:

$$
\begin{aligned}
|I_1^{o,C}| &\leqslant \int_{\widetilde{\Sigma}_p^o}\frac{|(C^0_{w^{\Sigma^o_\blacksquare}}\mathrm{I})(s)||w_+^{\Sigma^o_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi} \leqslant \frac{\|(C^0_{w^{\Sigma^o_\blacksquare}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)}\|w_+^{\Sigma^o_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}_p^o)}\\
&\leqslant \frac{\mathrm{const.}\,\|w_+^{\Sigma^o_\blacksquare}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\blacksquare)}(\|w_+^{\Sigma^o_\circlearrowright}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\circlearrowright)}+\|w_+^{\Sigma^o_\blacksquare}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\blacksquare)})}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}_p^o)}\\
&\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}\,\mathrm{dist}(z,\widetilde{\Sigma}_p^o)}\right)\left(\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\right)\\
&\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})\,\mathrm{dist}(z,\widetilde{\Sigma}_p^o)}\right),
\end{aligned}
$$

where, here and below, const. denotes some positive, $\mathcal{O}(1)$ constant; in going from the first to the second (resp., second to the third) line in the above asymptotic (as $n\to\infty$) estimation for $I_1^{o,C}$, one uses the fact that $\|(C^0_{w^{\Sigma^o_\blacksquare}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\circlearrowright)} \leqslant_{n\to\infty} \mathcal{O}(f(n)(n+1/2)^{-1}\mathrm{e}^{-(n+\frac{1}{2})c})$ and $\|(C^0_{w^{\Sigma^o_\blacksquare}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\blacksquare)} \leqslant_{n\to\infty} \mathcal{O}(f(n)(n+1/2)^{-1/2}\mathrm{e}^{-(n+\frac{1}{2})c})$ (resp., for $a,b>0$, $\sqrt{a^2+b^2}\leqslant\sqrt{a^2}+\sqrt{b^2}$) (facts used repeatedly below). One estimates the Cauchy part of $I_2^o$, denoted $I_2^{o,C}$, as follows:

$$
|I_2^{o,C}| \leqslant \int_{\widetilde{\Sigma}_p^o}\frac{|(C^0_{w^{\Sigma^o_\circlearrowright}}\mathrm{I})(s)||w_+^{\Sigma^o_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi} \leqslant \frac{\|(C^0_{w^{\Sigma^o_\circlearrowright}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)}\|w_+^{\Sigma^o_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}_p^o)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}_p^o)}
$$

$$
\begin{aligned}
&\leqslant \frac{\text{const.}\, \|w_{+}^{\Sigma^{o}_{\cup}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\cup})}\big(\|w_{+}^{\Sigma^{o}_{\cup}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\cup})}+\|w_{+}^{\Sigma^{o}_{\blacksquare}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\blacksquare})}\big)}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})} \\
&\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})}\right)\left(\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\right) \\
&\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})^{2}\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})}\right):
\end{aligned}
$$

in going from the second to the third line in the above asymptotic (as $n\to\infty$) estimation for $I_{2}^{o,C}$, one uses the fact that $\|(C^{0}_{w^{\Sigma^{o}_{\cup}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\cup})}\leqslant_{n\to\infty}\mathcal{O}(f(n)(n+1/2)^{-1})$ and $\|(C^{0}_{w^{\Sigma^{o}_{\cup}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\blacksquare})}\leqslant_{n\to\infty}\mathcal{O}(f(n)(n+1/2)^{-1})$. One estimates the Cauchy part of $I_{3}^{o}$, denoted $I_{3}^{o,C}$, as follows:

$$
\begin{aligned}
|I_{3}^{o,C}| &\leqslant \int_{\widetilde{\Sigma}^{o}_{p}}\frac{|((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(s)||w_{+}^{\Sigma^{o}_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi} \\
&\leqslant \frac{\|((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})} \\
&\leqslant \frac{\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})} \\
&\leqslant \frac{\text{const.}\,\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\blacksquare}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\blacksquare})}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})} \\
&\quad\times\left(\|w_{+}^{\Sigma^{o}_{\cup}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\cup})}+\|w_{+}^{\Sigma^{o}_{\blacksquare}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\blacksquare})}\right);
\end{aligned}
$$

using the fact that (cf. Proposition 5.2) $\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}=_{n\to\infty}\mathcal{O}(1)$ (via a Neuman series inversion argument, since $\|C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\leqslant_{n\to\infty}\mathcal{O}((n+1/2)^{-1/2}f(n)\mathrm{e}^{-(n+\frac{1}{2})c})$), one gets that

$$
\begin{aligned}
|I_{3}^{o,C}| &\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})}\right)\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\right) \\
&\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^{2}\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})}\right).
\end{aligned}
$$

One estimates the Cauchy part of $I_{4}^{o}$, denoted $I_{4}^{o,C}$, as follows:

$$
\begin{aligned}
|I_{4}^{o,C}| &\leqslant \int_{\widetilde{\Sigma}^{o}_{p}}\frac{|((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\cup}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\cup}}}(C^{0}_{w^{\Sigma^{o}_{\cup}}}\mathrm{I}))(s)||w_{+}^{\Sigma^{o}_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi} \\
&\leqslant \frac{\|((\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\cup}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\cup}}}(C^{0}_{w^{\Sigma^{o}_{\cup}}}\mathrm{I}))(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})} \\
&\leqslant \frac{\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\cup}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|C^{0}_{w^{\Sigma^{o}_{\cup}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|(C^{0}_{w^{\Sigma^{o}_{\cup}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})} \\
&\leqslant \frac{\text{const.}\,\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\cup}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|C^{0}_{w^{\Sigma^{o}_{\cup}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\cup}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\cup})}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^{o}_{p})}
\end{aligned}
$$

$$\times\left(\|w_{+}^{\Sigma^{o}_{\circlearrowright}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\circlearrowright})}+\|w_{+}^{\Sigma^{o}_{\blacksquare}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\blacksquare})}\right);$$

using the fact that (cf. Proposition 5.2) $\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}=_{n\to\infty}\mathcal{O}(1)$ (via a Neuman series inversion argument, since $\|C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}=_{n\to\infty}\mathcal{O}((n+1/2)^{-1}f(n))$), one gets that

$$\begin{aligned}|I^{o,C}_{4}| &\underset{n\to\infty}{\leqslant}\mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})\operatorname{dist}(z,\widetilde{\Sigma}^{o}_{p})}\right)\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\right)\\ &\underset{n\to\infty}{\leqslant}\mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})^{3}\operatorname{dist}(z,\widetilde{\Sigma}^{o}_{p})}\right).\end{aligned}$$

One estimates the Cauchy part of $I^{o}_{5}$, denoted $I^{o,C}_{5}$, as follows:

$$\begin{aligned}|I^{o,C}_{5}| \leqslant&\int_{\widetilde{\Sigma}^{o}_{p}}|((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\\ &\times\frac{C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(s)||w_{+}^{\Sigma^{o}_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi}\\ \leqslant&\|((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\\ &\times\frac{C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I}))(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}}{2\pi\operatorname{dist}(z,\widetilde{\Sigma}^{o}_{p})}\\ \leqslant&\|(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\\ &\times\frac{\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|(C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\mathrm{I})(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^{o}_{p})}}{2\pi\operatorname{dist}(z,\widetilde{\Sigma}^{o}_{p})}\\ \leqslant&\|(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\\ &\times\text{const.}\,\|(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\|w_{+}^{\Sigma^{o}_{\blacksquare}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\blacksquare})}\\ &\times\frac{(\|w_{+}^{\Sigma^{o}_{\circlearrowright}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\circlearrowright})}+\|w_{+}^{\Sigma^{o}_{\blacksquare}}(\cdot)\|_{\mathcal{L}^{2}_{\mathrm{M}_2(\mathbb{C})}(\Sigma^{o}_{\blacksquare})})}{2\pi\operatorname{dist}(z,\widetilde{\Sigma}^{o}_{p})};\end{aligned}$$

using the fact that (cf. Proposition 5.2) $\|(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}=_{n\to\infty}\mathcal{O}(1)$ (via a Neuman series inversion argument, since $\|C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}\leqslant_{n\to\infty}\mathcal{O}((n+\frac{1}{2})^{-1/2}f(n)\mathrm{e}^{-(n+\frac{1}{2})c})$ and $\|C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^{o}_{p})}=_{n\to\infty}\mathcal{O}((n+1/2)^{-1}f(n))$), one gets that

$$\begin{aligned}|I^{o,C}_{5}| &\underset{n\to\infty}{\leqslant}\mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})\operatorname{dist}(z,\widetilde{\Sigma}^{o}_{p})}\right)\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\right)\\ &\underset{n\to\infty}{\leqslant}\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^{2}\operatorname{dist}(z,\widetilde{\Sigma}^{o}_{p})}\right).\end{aligned}$$

One estimates the Cauchy part of $I^{o}_{6}$, denoted $I^{o,C}_{6}$, as follows:

$$\begin{aligned}|I^{o,C}_{6}| \leqslant&\int_{\widetilde{\Sigma}^{o}_{p}}|((\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}})^{-1}\\ &\times\frac{C^{0}_{w^{\Sigma^{o}_{\blacksquare}}}(C^{0}_{w^{\Sigma^{o}_{\circlearrowright}}}\mathrm{I}))(s)||w_{+}^{\Sigma^{o}_{\mathcal{R}}}(s)|}{|s-z|}\frac{|\mathrm{d}s|}{2\pi}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|((\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}\\
&\times \frac{C^0_{w^{\Sigma^o_\blacksquare}}(C^0_{w^{\Sigma^o_\circlearrowright}}\mathrm{I}))(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\|w_+^{\Sigma^o_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\\
&\leqslant \|(\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\\
&\times \frac{\|(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|C^0_{w^{\Sigma^o_\blacksquare}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|(C^0_{w^{\Sigma^o_\circlearrowright}}\mathrm{I})(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}\|w_+^{\Sigma^o_{\mathcal{R}}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Sigma}^o_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\\
&\leqslant \|(\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\\
&\times \text{const.}\,\|(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|C^0_{w^{\Sigma^o_\blacksquare}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|w_+^{\Sigma^o_\circlearrowright}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\circlearrowright)}\\
&\times \frac{(\|w_+^{\Sigma^o_\circlearrowright}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\circlearrowright)}+\|w_+^{\Sigma^o_\blacksquare}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\blacksquare)})}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)};
\end{aligned}
$$

using the fact that (cf. Proposition 5.2) $\|(\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)} =_{n\to\infty} \mathcal{O}(1)$ (via a Neuman series inversion argument, since $\|C^0_{w^{\Sigma^o_\blacksquare}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)} =_{n\to\infty} \mathcal{O}((n+\frac{1}{2})^{-1/2}f(n)\mathrm{e}^{-(n+\frac{1}{2})c})$ and $\|C^0_{w^{\Sigma^o_\circlearrowright}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)} =_{n\to\infty} \mathcal{O}((n+1/2)^{-1}f(n)))$, one gets that

$$
\begin{aligned}
|I_6^{o,C}| &\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\right)\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\right)\\
&\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^2\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\right).
\end{aligned}
$$

One estimates, succinctly, the Cauchy part of $I^o_7$, denoted $I_7^{o,C}$, as follows:

$$
\begin{aligned}
|I_7^{o,C}| &\leqslant \int_{\widetilde{\Sigma}^o_p}|((\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}C^0_{w^{\Sigma^o_\blacksquare}}C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\\
&\times \frac{C^0_{w^{\Sigma^o_\circlearrowright}}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}C^0_{w^{\Sigma^o_\blacksquare}}(C^0_{w^{\Sigma^o_\blacksquare}}\mathrm{I}))(s)\|w_+^{\Sigma^o_{\mathcal{R}}}(s)|}{|s-z|}\,\frac{|\mathrm{d}s|}{2\pi}\\
&\leqslant \frac{\|(\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}C^0_{w^{\Sigma^o_\blacksquare}}C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\\
&\times \|(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|C^0_{w^{\Sigma^o_\circlearrowright}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|C^0_{w^{\Sigma^o_\blacksquare}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\\
&\times \text{const.}\,\|w_+^{\Sigma^o_\blacksquare}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\blacksquare)}\left(\|w_+^{\Sigma^o_\blacksquare}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\blacksquare)}+\|w_+^{\Sigma^o_\circlearrowright}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_\circlearrowright)}\right);
\end{aligned}
$$

using the fact that (established above) $\|(\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}C^0_{w^{\Sigma^o_\blacksquare}}C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)} =_{n\to\infty} \mathcal{O}(1)$, one gets that

$$
\begin{aligned}
|I_7^{o,C}| &\underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\right)\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)\right.\\
&\quad\left.+\ \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\right) \underset{n\to\infty}{\leqslant} \mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^3\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\right).
\end{aligned}
$$

One estimates, succinctly, the Cauchy part of $I^o_8$, denoted $I_8^{o,C}$, as follows:

$$
|I_8^{o,C}| \leqslant \int_{\widetilde{\Sigma}^o_p}|((\mathbf{id}-(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}C^0_{w^{\Sigma^o_\circlearrowright}}C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\blacksquare}})^{-1}(\mathbf{id}-C^0_{w^{\Sigma^o_\circlearrowright}})^{-1}
$$

$$\times\ \frac{C^{0}_{w^{\Sigma^o_{\blacksquare}}}(\mathbf{id}-C^{0}_{w^{\Sigma^o_{\cup}}})^{-1}C^{0}_{w^{\Sigma^o_{\cup}}}(C^{0}_{w^{\Sigma^o_{\cup}}}\mathrm{I}))(s)\|w_+^{\Sigma^o_{\mathcal{R}}}(s)|}{|s-z|}\,\frac{|\mathrm{d}s|}{2\pi}$$

$$\leqslant\ \frac{\|(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^o_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^o_{\cup}}})^{-1}C^{0}_{w^{\Sigma^o_{\cup}}}C^{0}_{w^{\Sigma^o_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|(\mathbf{id}-C^{0}_{w^{\Sigma^o_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}}{2\pi\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}$$

$$\times\ \|(\mathbf{id}-C^{0}_{w^{\Sigma^o_{\cup}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|C^{0}_{w^{\Sigma^o_{\blacksquare}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|(\mathbf{id}-C^{0}_{w^{\Sigma^o_{\cup}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}\|C^{0}_{w^{\Sigma^o_{\cup}}}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}$$

$$\times\ \mathrm{const.}\,\|w_+^{\Sigma^o_{\cup}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_{\cup})}\Big(\|w_+^{\Sigma^o_{\blacksquare}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_{\blacksquare})}+\|w_+^{\Sigma^o_{\cup}}(\cdot)\|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma^o_{\cup})}\Big);$$

using the fact that (established above) $\|(\mathbf{id}-(\mathbf{id}-C^{0}_{w^{\Sigma^o_{\blacksquare}}})^{-1}(\mathbf{id}-C^{0}_{w^{\Sigma^o_{\cup}}})^{-1}C^{0}_{w^{\Sigma^o_{\cup}}}C^{0}_{w^{\Sigma^o_{\blacksquare}}})^{-1}\|_{\mathcal{N}_2(\widetilde{\Sigma}^o_p)}=_{n\to\infty}\mathcal{O}(1)$, one gets that

$$|I_8^{o,\mathcal{C}}|\underset{n\to\infty}{\leqslant}\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\right)\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)\left(\mathcal{O}\left(\frac{f(n)}{n+\frac{1}{2}}\right)+\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{\sqrt{n+\frac{1}{2}}}\right)\right)$$
$$\underset{n\to\infty}{\leqslant}\mathcal{O}\left(\frac{f(n)\mathrm{e}^{-(n+\frac{1}{2})c}}{(n+\frac{1}{2})^3\,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)}\right).$$

Analogously, estimating (as $n\to\infty$) the non-Cauchy contributions (corresponding to the kernel $\frac{1}{s}$) of the terms on the right-hand side of Equation (5.2) which, too, are $\mathcal{O}((n+1/2)^{-2})$, and gathering all derived (upper) bounds, one arrives at the result stated in the Lemma. □

**Lemma 5.3.** *Let $\mathcal{R}^o\colon\mathbb{C}\setminus\widetilde{\Sigma}^o_p\to\mathrm{SL}_2(\mathbb{C})$ be the solution of the* RHP $(\mathcal{R}^o(z),v^o_{\mathcal{R}}(z),\widetilde{\Sigma}^o_p)$ *formulated in Proposition* 5.2 *with the $n\to\infty$ integral representation given in Lemma* 5.2*. Then, uniformly for compact subsets of $\mathbb{C}\setminus\widetilde{\Sigma}^o_p\ni z$,*

$$\mathcal{R}^o(z)\underset{\substack{n\to\infty\\ z\in\mathbb{C}\setminus\widetilde{\Sigma}^o_p}}{=}\mathrm{I}+\frac{1}{(n+\frac{1}{2})}\Big(\mathcal{R}^o_0(z)-\widetilde{\mathcal{R}}^o_0(z)\Big)+\mathcal{O}\left(\frac{f(z;n)}{(n+\frac{1}{2})^2}\right),$$

*where $\mathcal{R}^o_0(z)$ is defined in Theorem* 2.3.1*, Equations* (2.23)–(2.57)*, $\widetilde{\mathcal{R}}^o_0(z)$ is defined in Theorem* 2.3.1*, Equations* (2.14)–(2.20) *and* (2.70)–(2.74)*, and $f(z;n)$, where the $n$-dependence arises due to the $n$-dependence of the associated Riemann theta functions, is a bounded (with respect to $z$ and $n$),* $\mathrm{GL}_2(\mathbb{C})$*-valued function which is analytic (with respect to $z$) for $z\in\mathbb{C}\setminus\widetilde{\Sigma}^o_p$, and $(f(\cdot;n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$.*

**Remark 5.1.** Note from the formulation of Lemma 5.3 above that (cf. Theorem 2.3.1, Equations (2.24)–(2.27)), for $j=1,\dots,N+1$, $\mathrm{tr}(\mathcal{A}^o(a^o_j))=\mathrm{tr}(\mathcal{A}^o(b^o_{j-1}))=\mathrm{tr}(\mathcal{B}^o(a^o_j))=\mathrm{tr}(\mathcal{B}^o(b^o_{j-1}))=0$. ■

*Proof.* Recall the integral representation for $\mathcal{R}^o\colon\mathbb{C}\setminus\widetilde{\Sigma}^o_p\to\mathrm{SL}_2(\mathbb{C})$ given in Lemma 5.2:

$$\mathcal{R}^o(z)\underset{n\to\infty}{=}\mathrm{I}+\int_{\Sigma^o_{\cup}}\frac{zw_+^{\Sigma^o_{\cup}}(s)}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})^2\min\{1,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)\}}\right),\quad z\in\mathbb{C}\setminus\widetilde{\Sigma}^o_p,$$

where $\Sigma^o_{\cup}:=\cup_{j=1}^{N+1}(\partial\mathbb{U}^o_{\delta_{b_{j-1}}}\cup\partial\mathbb{U}^o_{\delta_{a_j}})$, and $(f(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$. Recalling that the radii of the open discs $\mathbb{U}^o_{\delta_{b_{j-1}}},\mathbb{U}^o_{\delta_{a_j}}$, $j=1,\dots,N+1$, are chosen, amongst other factors (cf. Lemmae 4.6 and 4.7), such that $\mathbb{U}^o_{\delta_{b_{j-1}}}\cap\mathbb{U}^o_{\delta_{a_k}}=\varnothing$, $j,k=1,\dots,N+1$, it follows from the above integral representation that

$$\mathcal{R}^o(z)\underset{n\to\infty}{=}\mathrm{I}-\sum_{j=1}^{N+1}\left(\oint_{\partial\mathbb{U}^o_{\delta_{b_{j-1}}}}+\oint_{\partial\mathbb{U}^o_{\delta_{a_j}}}\right)\frac{zw_+^{\Sigma^o_{\cup}}(s)}{s(s-z)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}+\mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})^2\min\{1,\mathrm{dist}(z,\widetilde{\Sigma}^o_p)\}}\right),\quad z\in\mathbb{C}\setminus\widetilde{\Sigma}^o_p,$$

where $\oint_{\partial\mathbb{U}^o_{\delta_{b_{j-1}}}},\oint_{\partial\mathbb{U}^o_{\delta_{a_j}}}$, $j=1,\dots,N+1$, are counter-clockwise-oriented, closed (contour) integrals (Figure 10) about the end-points of the support of the 'odd' equilibrium measure, $\{b^o_{j-1},a^o_j\}_{j=1}^{N+1}$. Noting the

partial fraction decomposition $\frac{z}{s(s-z)}=-\frac{1}{s}+\frac{1}{s-z}$, the evaluation of these $4(N+1)$ contour integrals requires the application of the Cauchy and Residue Theorems; and, since the evaluation of the respective integrals entails analogous calculations, consider, say, and without loss of generality, the evaluation of the integrals corresponding to the (standard) Cauchy kernel, $\frac{1}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$, about the right-most end-points $a_j^o$, $j=1,\dots,N$, namely:

$$\oint_{\partial\mathbb{U}^o_{\delta_{a_j}}}\frac{w_+^{\Sigma^o_{\circlearrowright}}(s)}{s-z}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}},\qquad j=1,\dots,N.$$

Recalling from Lemma 4.7 that $\xi^o_{a_j}(z)=(z-a_j^o)^{3/2}G^o_{a_j}(z)$, $z\in\mathbb{U}^o_{\delta_{a_j}}\setminus(-\infty,a_j^o)$, $j=1,\dots,N$, it follows from item (5) of Proposition 5.1 that, since $w_+^{\Sigma^o_{\circlearrowright}}(z)=v^o_{\mathcal{R}}(z)-\mathrm{I}$, for $j=1,\dots,N$,

$$\begin{aligned}w_+^{\Sigma^o_{\circlearrowright}}(z)\underset{\substack{n\to\infty\\ z\in\mathbb{C}_\pm\cap\partial\mathbb{U}^o_{\delta_{a_j}}}}{=}&\,\frac{1}{(n+\frac{1}{2})(z-a_j^o)^{3/2}G^o_{a_j}(z)}\overset{o}{\mathfrak{M}}^{\infty}(z)\begin{pmatrix}\mp(s_1+t_1) & \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o}\\ \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_j^o} & \pm(s_1+t_1)\end{pmatrix}\\ &\times(\overset{o}{\mathfrak{M}}^{\infty}(z))^{-1}+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2(z-a_j^o)^3(G^o_{a_j}(z))^2}\overset{o}{\mathfrak{M}}^{\infty}(z)f^o_{a_j}(n)(\overset{o}{\mathfrak{M}}^{\infty}(z))^{-1}\right),\end{aligned}$$

where $\overset{o}{\mathfrak{M}}^{\infty}(z)$ and $\Omega_j^o$ are defined in Lemma 4.5, and $(f^o_{a_j}(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$. A matrix-multiplication argument shows that $\overset{o}{\mathfrak{M}}^{\infty}(z)\begin{pmatrix}\mp(s_1+t_1) & \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o}\\ \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_j^o} & \pm(s_1+t_1)\end{pmatrix}(\overset{o}{\mathfrak{M}}^{\infty}(z))^{-1}$ is given by

$$\begin{pmatrix}\boxed{\begin{array}{r}\mp\frac{1}{4}(s_1+t_1)\left(\frac{(\gamma^o(z))^2+(\gamma^o(0))^2}{\gamma^o(0)\gamma^o(z)}\right)^2\mathfrak{m}^o_{11}(z)\mathfrak{m}^o_{22}(z)\\ \mp\frac{1}{4}(s_1+t_1)\left(\frac{(\gamma^o(z))^2-(\gamma^o(0))^2}{\gamma^o(0)\gamma^o(z)}\right)^2\mathfrak{m}^o_{12}(z)\mathfrak{m}^o_{21}(z)\\ \mp\frac{1}{4}(s_1-t_1)\left(\frac{(\gamma^o(z))^4-(\gamma^o(0))^4}{(\gamma^o(0)\gamma^o(z))^2}\right)\frac{\mathfrak{m}^o_{11}(z)\mathfrak{m}^o_{21}(z)}{\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}\\ \mp\frac{1}{4}(s_1-t_1)\left(\frac{(\gamma^o(z))^4-(\gamma^o(0))^4}{(\gamma^o(0)\gamma^o(z))^2}\right)\frac{\mathfrak{m}^o_{12}(z)\mathfrak{m}^o_{22}(z)}{\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}\end{array}} & \boxed{\begin{array}{r}\pm\frac{\mathrm{i}}{2}(s_1+t_1)\left(\frac{(\gamma^o(z))^4-(\gamma^o(0))^4}{(\gamma^o(0)\gamma^o(z))^2}\right)\mathfrak{m}^o_{11}(z)\mathfrak{m}^o_{12}(z)\\ \pm\frac{\mathrm{i}}{4}(s_1-t_1)\left(\frac{(\gamma^o(z))^2+(\gamma^o(0))^2}{\gamma^o(0)\gamma^o(z)}\right)^2\frac{(\mathfrak{m}^o_{11}(z))^2}{\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}\\ \pm\frac{\mathrm{i}}{4}(s_1-t_1)\left(\frac{(\gamma^o(z))^2-(\gamma^o(0))^2}{\gamma^o(0)\gamma^o(z)}\right)^2\frac{(\mathfrak{m}^o_{12}(z))^2}{\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}\end{array}}\\ \boxed{\begin{array}{r}\pm\frac{\mathrm{i}}{2}(s_1+t_1)\left(\frac{(\gamma^o(z))^4-(\gamma^o(0))^4}{(\gamma^o(0)\gamma^o(z))^2}\right)\mathfrak{m}^o_{21}(z)\mathfrak{m}^o_{22}(z)\\ \pm\frac{\mathrm{i}}{4}(s_1-t_1)\left(\frac{(\gamma^o(z))^2-(\gamma^o(0))^2}{\gamma^o(0)\gamma^o(z)}\right)^2\frac{(\mathfrak{m}^o_{21}(z))^2}{\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}\\ \pm\frac{\mathrm{i}}{4}(s_1-t_1)\left(\frac{(\gamma^o(z))^2+(\gamma^o(0))^2}{\gamma^o(0)\gamma^o(z)}\right)^2\frac{(\mathfrak{m}^o_{22}(z))^2}{\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}\end{array}} & \boxed{\begin{array}{r}\pm\frac{1}{4}(s_1+t_1)\left(\frac{(\gamma^o(z))^2+(\gamma^o(0))^2}{\gamma^o(0)\gamma^o(z)}\right)^2\mathfrak{m}^o_{11}(z)\mathfrak{m}^o_{22}(z)\\ \pm\frac{1}{4}(s_1+t_1)\left(\frac{(\gamma^o(z))^2-(\gamma^o(0))^2}{\gamma^o(0)\gamma^o(z)}\right)^2\mathfrak{m}^o_{12}(z)\mathfrak{m}^o_{21}(z)\\ \pm\frac{1}{4}(s_1-t_1)\left(\frac{(\gamma^o(z))^4-(\gamma^o(0))^4}{(\gamma^o(0)\gamma^o(z))^2}\right)\frac{\mathfrak{m}^o_{11}(z)\mathfrak{m}^o_{21}(z)}{\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}\\ \pm\frac{1}{4}(s_1-t_1)\left(\frac{(\gamma^o(z))^4-(\gamma^o(0))^4}{(\gamma^o(0)\gamma^o(z))^2}\right)\frac{\mathfrak{m}^o_{12}(z)\mathfrak{m}^o_{22}(z)}{\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}\end{array}}\end{pmatrix},$$

where $s_1$ and $t_1$ are given in Theorem 2.3.1, Equations (2.28), $\gamma^o(z)$ and $\gamma^o(0)$ are defined in Lemma 4.4, and $\mathfrak{m}^o_{kl}(z)$, $k,l=1,2$, are defined in Theorem 2.3.1, Equations (2.17)–(2.20). Recall that, for $j=1,\dots,N$, $\omega_j^o=\sum_{k=1}^N c^o_{jk}(R_o(z))^{-1/2}z^{N-k}\,\mathrm{d}z$, where $c^o_{jk}$, $j,k=1,\dots,N$, are obtained from Equations (O1) and (O2), and (the multi-valued function) $(R_o(z))^{1/2}$ is defined in Theorem 2.3.1, Equation (2.8). One shows that

$$\omega_m^o\underset{\substack{z\to a_j^o\\ j=1,\dots,N}}{=}\frac{(\mathfrak{f}_o(a_j^o))^{-1}}{\sqrt{z-a_j^o}}\left(\mathfrak{p}^{\natural}_m(a_j^o)+\mathfrak{q}^{\natural}_m(a_j^o)(z-a_j^o)+\mathfrak{r}^{\natural}_m(a_j^o)(z-a_j^o)^2+\mathcal{O}((z-a_j^o)^3)\right)\mathrm{d}z,\qquad m=1,\dots,N,$$

where

$$\mathfrak{f}_o(\xi)=(-1)^{N-j+1}\left((a^o_{N+1}-\xi)(\xi-b_0^o)(b_j^o-\xi)\prod_{k=1}^{j-1}(\xi-b_k^o)(\xi-a_k^o)\prod_{l=j+1}^{N}(b_l^o-\xi)(a_l^o-\xi)\right)^{1/2},$$

$$\mathfrak{p}^{\natural}_m(\xi)=\sum_{k=1}^N c^o_{mk}\xi^{N-k},\qquad\qquad \mathfrak{q}^{\natural}_m(\xi)=\sum_{k=1}^N c^o_{mk}\xi^{N-k-1}\left(N-k-\frac{\xi\mathfrak{f}_o'(\xi)}{\mathfrak{f}_o(\xi)}\right),$$

$$\mathfrak{r}_m^{\natural}(\xi)=\sum_{k=1}^{N}c_{mk}^{o}\xi^{N-k-2}\left(\frac{(N-k)(N-k-1)}{2}-\frac{(N-k)\xi\mathfrak{f}_o'(\xi)}{\mathfrak{f}_o(\xi)}+\xi^2\left(\left(\frac{\mathfrak{f}_o'(\xi)}{\mathfrak{f}_o(\xi)}\right)^2-\frac{\mathfrak{f}_o''(\xi)}{2\mathfrak{f}_o(\xi)}\right)\right),$$

with $(-1)^{-N+j-1}\mathfrak{f}_o(a_j^o)>0$,

$$\mathfrak{f}_o'(\xi)=\frac{1}{2}\mathfrak{f}_o(\xi)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{\xi-b_k^o}+\frac{1}{\xi-a_k^o}\right)+\frac{1}{\xi-b_j^o}+\frac{1}{\xi-a_{N+1}^o}+\frac{1}{\xi-b_0^o}\right),$$

$$\begin{aligned}\mathfrak{f}_o''(\xi)=&-\frac{1}{2}\mathfrak{f}_o(\xi)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{(\xi-b_k^o)^2}+\frac{1}{(\xi-a_k^o)^2}\right)+\frac{1}{(\xi-b_j^o)^2}+\frac{1}{(\xi-a_{N+1}^o)^2}+\frac{1}{(\xi-b_0^o)^2}\right)\\&+\frac{1}{4}\mathfrak{f}_o(\xi)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{\xi-b_k^o}+\frac{1}{\xi-a_k^o}\right)+\frac{1}{\xi-b_j^o}+\frac{1}{\xi-a_{N+1}^o}+\frac{1}{\xi-b_0^o}\right)^2.\end{aligned}$$

Recall (cf. Lemma 4.5), also, that $\boldsymbol{u}^o\equiv\int_{a_{N+1}^o}^{z}\boldsymbol{\omega}^o$ $(\in\mathrm{Jac}(\mathcal{Y}_o))$, where $\equiv$ denotes congruence modulo the period lattice, with $\boldsymbol{\omega}^o:=(\omega_1^o,\omega_2^o,\dots,\omega_N^o)$; hence, via the above expansion (as $z\to a_j^o$, $j=1,\dots,N$) for $\omega_m^o$, $m=1,\dots,N$, one arrives at

$$\int_{a_j^o}^{z}\omega_m^o\underset{\substack{z\to a_j^o\\j=1,\dots,N}}{\equiv}\frac{2\mathfrak{p}_m^{\natural}(a_j^o)}{\mathfrak{f}_o(a_j^o)}(z-a_j^o)^{1/2}+\frac{2\mathfrak{q}_m^{\natural}(a_j^o)}{3\mathfrak{f}_o(a_j^o)}(z-a_j^o)^{3/2}+\frac{2\mathfrak{r}_m^{\natural}(a_j^o)}{5\mathfrak{f}_o(a_j^o)}(z-a_j^o)^{5/2}+\mathcal{O}((z-a_j^o)^{7/2}).$$

From the definition of $\mathfrak{m}_{kl}^o(z)$, $k,l=1,2$, given in Theorem 2.3.1, Equations (2.17)–(2.20), the definition of the 'odd' Riemann theta function given by Equation (2.1), and recalling that $\mathfrak{m}_{kl}^o(z)$, $k,l=1,2$, satisfy the jump relation (cf. Lemma 4.5) $\mathfrak{m}_+^o(z)=\mathfrak{m}_-^o(z)(\exp(-\mathrm{i}(n+\frac{1}{2})\Omega_j^o)\sigma_-+\exp(\mathrm{i}(n+\frac{1}{2})\Omega_j^o)\sigma_+)$, via the above asymptotic expansion (as $z\to a_j^o$, $j=1,\dots,N$) for $\int_{a_j^o}^{z}\omega_m^o$, $m=1,\dots,N$, one arrives at

$$\begin{aligned}\mathfrak{m}_{11}^o(z)\underset{\substack{z\to a_j^o\\j=1,\dots,N}}{=}&\varkappa_1^o(a_j^o)\Big(1+\mathrm{i}\aleph_1^1(a_j^o)(z-a_j^o)^{1/2}+\daleth_1^1(a_j^o)(z-a_j^o)+\mathrm{i}\beth_1^1(a_j^o)(z-a_j^o)^{3/2}+\gimel_1^1(a_j^o)(z-a_j^o)^2\\&+\mathcal{O}((z-a_j^o)^{5/2})\Big),\\
\mathfrak{m}_{12}^o(z)\underset{\substack{z\to a_j^o\\j=1,\dots,N}}{=}&\varkappa_1^o(a_j^o)\Big(1-\mathrm{i}\aleph_1^{-1}(a_j^o)(z-a_j^o)^{1/2}+\daleth_1^{-1}(a_j^o)(z-a_j^o)-\mathrm{i}\beth_1^{-1}(a_j^o)(z-a_j^o)^{3/2}+\gimel_1^{-1}(a_j^o)(z-a_j^o)^2\\&+\mathcal{O}((z-a_j^o)^{5/2})\Big)\exp\left(\mathrm{i}\left(n+\frac{1}{2}\right)\Omega_j^o\right),\\
\mathfrak{m}_{21}^o(z)\underset{\substack{z\to a_j^o\\j=1,\dots,N}}{=}&\varkappa_2^o(a_j^o)\Big(1+\mathrm{i}\aleph_{-1}^1(a_j^o)(z-a_j^o)^{1/2}+\daleth_{-1}^1(a_j^o)(z-a_j^o)+\mathrm{i}\beth_{-1}^1(a_j^o)(z-a_j^o)^{3/2}+\gimel_{-1}^1(a_j^o)(z-a_j^o)^2\\&+\mathcal{O}((z-a_j^o)^{5/2})\Big),\\
\mathfrak{m}_{22}^o(z)\underset{\substack{z\to a_j^o\\j=1,\dots,N}}{=}&\varkappa_2^o(a_j^o)\Big(1-\mathrm{i}\aleph_{-1}^{-1}(a_j^o)(z-a_j^o)^{1/2}+\daleth_{-1}^{-1}(a_j^o)(z-a_j^o)-\mathrm{i}\beth_{-1}^{-1}(a_j^o)(z-a_j^o)^{3/2}+\gimel_{-1}^{-1}(a_j^o)(z-a_j^o)^2\\&+\mathcal{O}((z-a_j^o)^{5/2})\Big)\exp\left(\mathrm{i}\left(n+\frac{1}{2}\right)\Omega_j^o\right),\end{aligned}$$

where, for $\varepsilon_1,\varepsilon_2=\pm1$,

$$\varkappa_1^o(\xi)=\frac{1}{\mathbb{E}}\frac{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)+\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(\xi)+\boldsymbol{d}_o)},$$

$$\varkappa_2^o(\xi)=\mathbb{E}\frac{\boldsymbol{\theta}^o(-\boldsymbol{u}_+^o(0)-\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(-\boldsymbol{u}_+^o(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)\boldsymbol{\theta}^o(\boldsymbol{u}_+^o(\xi)-\boldsymbol{d}_o)},$$

$$
\begin{aligned}
\aleph^{\varepsilon_1}_{\varepsilon_2}(\xi) =& -\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)}+\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)},\\
\daleth^{\varepsilon_1}_{\varepsilon_2}(\xi) =& -\frac{\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)}+\frac{\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}-\left(\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)}\right)^{2}\\
&+\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)},\\
\beth^{\varepsilon_1}_{\varepsilon_2}(\xi) =& -\frac{\mathfrak{w}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)}+\frac{\mathfrak{w}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}+\frac{2\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{(\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o))^{2}}\\
&-\frac{\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}+\left(\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)}\right)^{3}\\
&-\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}-\frac{(\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi))^{2}}{(\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o))^{2}}\\
&\times\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)},\\
\gimel^{\varepsilon_1}_{\varepsilon_2}(\xi) =& -\frac{\mathfrak{z}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)}+\frac{\mathfrak{z}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}+\left(\frac{\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)}\right)^{2}\\
&-\frac{\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}-\frac{2\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{w}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{(\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o))^{2}}\\
&+\frac{\mathfrak{w}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}+\frac{3(\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi))^{2}\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{(\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o))^{3}}\\
&+\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{w}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}+\left(\frac{\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o)}\right)^{4}\\
&-\frac{2\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi)\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{(\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o))^{2}\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}-\frac{(\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi))^{2}}{(\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o))^{2}}\\
&\times\frac{\mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)}-\frac{(\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\mathbf{0};\xi))^{3}\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)}{(\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)+\varepsilon_2\boldsymbol{d}_o))^{3}\boldsymbol{\theta}^{o}(\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)},
\end{aligned}
$$

with $\mathbf{0}:=(0,0,\dots,0)^{\mathrm{T}}$ $(\in\mathbb{R}^{N})$,

$$
\mathfrak{u}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi):=2\pi\overset{o}{\Lambda}{}^{1}_{0}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi),\qquad \mathfrak{v}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi):=-2\pi^{2}\overset{o}{\Lambda}{}^{2}_{0}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi),
$$

$$
\mathfrak{w}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi):=2\pi\left(\overset{o}{\Lambda}{}^{0}_{1}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)-\frac{2\pi^{2}}{3}\overset{o}{\Lambda}{}^{3}_{0}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)\right),
$$

$$
\mathfrak{z}^{o}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi):=-(2\pi)^{2}\left(\overset{o}{\Lambda}{}^{1}_{1}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)-\frac{\pi^{2}}{6}\overset{o}{\Lambda}{}^{4}_{0}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)\right),
$$

$$
\overset{o}{\Lambda}{}^{j_1}_{j_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^{o};\xi)=\sum_{m\in\mathbb{Z}^{N}}(\mathfrak{r}_o(\xi))^{j_1}(\mathfrak{s}_o(\xi))^{j_2}\,\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}^{o}_{+}(\xi)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_2\boldsymbol{d}_o)+\pi\mathrm{i}(m,\boldsymbol{\tau}^{o}m)},
$$

$$
\mathfrak{r}_o(\xi):=\frac{2(m,\overset{\rightarrow o}{\pm}_{1}(\xi))}{\mathfrak{f}_o(\xi)},\qquad \mathfrak{s}_o(\xi):=\frac{2(m,\overset{\rightarrow o}{\pm}_{2}(\xi))}{3\mathfrak{f}_o(\xi)},
$$

$$
\overset{\rightarrow o}{\pm}_{1}(\xi)=\left(\mathfrak{p}^{\natural}_{1}(\xi),\mathfrak{p}^{\natural}_{2}(\xi),\dots,\mathfrak{p}^{\natural}_{N}(\xi)\right),\qquad \overset{\rightarrow o}{\pm}_{2}(\xi)=\left(\mathfrak{q}^{\natural}_{1}(\xi),\mathfrak{q}^{\natural}_{2}(\xi),\dots,\mathfrak{q}^{\natural}_{N}(\xi)\right).
$$

Recall the definition of $\gamma^{o}(z)$ given in Lemma 4.4: a careful analysis of the branch cuts shows that, for $j=1,\dots,N$,

$$
(\gamma^{o}(z))^{2}\underset{z\in\mathbb{C}_{\pm}}{=}\pm\frac{\left((z-b^{o}_{j})\prod_{\substack{k=1\\k\neq j}}^{N}\left(\frac{z-b^{o}_{k}}{z-a^{o}_{k}}\right)\left(\frac{z-b^{o}_{0}}{z-a^{o}_{N+1}}\right)\right)^{1/2}}{\sqrt{z-a^{o}_{j}}}
$$

$$\underset{\mathbb{C}_{\pm}\ni z\to a_j^o}{=} \pm \frac{\left(Q_0^o(a_j^o)+Q_1^o(a_j^o)(z-a_j^o)+\frac{1}{2}Q_2^o(a_j^o)(z-a_j^o)^2+\mathcal{O}((z-a_j^o)^3)\right)}{\sqrt{z-a_j^o}},$$

where $Q_0^o(a_j^o)$, $Q_1^o(a_j^o)$, $j=1,\dots,N$, are given in Theorem 2.3.1, Equations (2.35) and (2.36), and

$$\begin{aligned} Q_2^o(a_j^o) =& -\frac{1}{2}Q_0^o(a_j^o)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{(a_j^o-b_k^o)^2}-\frac{1}{(a_j^o-a_k^o)^2}\right)+\frac{1}{(a_j^o-b_0^o)^2}-\frac{1}{(a_j^o-a_{N+1}^o)^2}+\frac{1}{(a_j^o-b_j^o)^2}\right) \\ &+\frac{1}{4}Q_0^o(a_j^o)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{a_j^o-b_k^o}-\frac{1}{a_j^o-a_k^o}\right)+\frac{1}{a_j^o-b_0^o}-\frac{1}{a_j^o-a_{N+1}^o}+\frac{1}{a_j^o-b_j^o}\right)^2, \quad j=1,\dots,N. \end{aligned}$$

Recall the formula above for $\overset{o}{\mathfrak{M}}{}^{\infty}(z)\begin{pmatrix} \mp(s_1+t_1) & \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o} \\ \pm\mathrm{i}(s_1-t_1)\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\Omega_j^o} & \pm(s_1+t_1) \end{pmatrix}(\overset{o}{\mathfrak{M}}{}^{\infty}(z))^{-1}$. Substituting the above expansions (as $z\to a_j^o$, $j=1,\dots,N$) into this formula, equating coefficients of like powers of $(z-a_j^o)^{-p/2}(\widetilde{n}G_{a_j}^o(z))^{-1}$, $p\in\{4,3,2,1,0\}$, with $\widetilde{n}:=n+\frac{1}{2}$, and considering, say, the $(1\,1)$-element of the resulting (asymptotic) expansions, one arrives at (modulo a minus sign, this result is equally applicable to the $(2\,2)$-element, since $\mathrm{tr}(w_{+}^{\Sigma_{\circlearrowright}^o}(z))=0$), up to terms that are $\mathcal{O}((\widetilde{n}^2(z-a_j^o)^3(G_{a_j}^o(z))^2)^{-1}\overset{o}{\mathfrak{M}}{}^{\infty}(z)f_{a_j}^o(n)(\overset{o}{\mathfrak{M}}{}^{\infty}(z))^{-1})$, upon setting, for economy of notation, $Q_q^o(a_j^o)=:Q_q$, $q=0,1,2$, $\gamma^o(0)=:\gamma^o$, $\varkappa_1^o(a_j^o)=:\varkappa_1^o$, $\varkappa_2^o(a_j^o)=:\varkappa_2^o$, $\aleph_{\varepsilon_2}^{\varepsilon_1}(a_j^o)=:\aleph_{\varepsilon_2}^{\varepsilon_1}$, $\daleth_{\varepsilon_2}^{\varepsilon_1}(a_j^o)=:\daleth_{\varepsilon_2}^{\varepsilon_1}$, $\beth_{\varepsilon_2}^{\varepsilon_1}(a_j^o)=:\beth_{\varepsilon_2}^{\varepsilon_1}$, and $\gimel_{\varepsilon_2}^{\varepsilon_1}(a_j^o)=:\gimel_{\varepsilon_2}^{\varepsilon_1}$:

$$\mathcal{O}\left(\frac{(z-a_j^o)^{-2}\,\mathrm{e}^{\mathrm{i}\widetilde{n}\Omega_j^o}}{\widetilde{n}G_{a_j}^o(z)}\right): -\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^oQ_0}{4(\gamma^o)^2}-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^oQ_0}{4(\gamma^o)^2}-\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^oQ_0}{4(\gamma^o)^2}-\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^oQ_0}{4(\gamma^o)^2};$$

$$\begin{aligned} \mathcal{O}\left(\frac{(z-a_j^o)^{-3/2}\,\mathrm{e}^{\mathrm{i}\widetilde{n}\Omega_j^o}}{\widetilde{n}G_{a_j}^o(z)}\right): &-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^oQ_0(\aleph_1^1-\aleph_{-1}^{-1})}{4(\gamma^o)^2}-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^oQ_0(\aleph_{-1}^1-\aleph_1^{-1})}{4(\gamma^o)^2} \\ &-\frac{\mathrm{i}(s_1-t_1)\varkappa_1^o\varkappa_2^oQ_0(\aleph_{-1}^1+\aleph_1^1)}{4(\gamma^o)^2}+\frac{\mathrm{i}(s_1-t_1)\varkappa_1^o\varkappa_2^oQ_0(\aleph_{-1}^{-1}+\aleph_1^{-1})}{4(\gamma^o)^2} \\ &-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o}{2}+\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o}{2}; \end{aligned}$$

$$\begin{aligned} \mathcal{O}\left(\frac{(z-a_j^o)^{-1}\,\mathrm{e}^{\mathrm{i}\widetilde{n}\Omega_j^o}}{\widetilde{n}G_{a_j}^o(z)}\right): &-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\left(Q_1+Q_0\left(\daleth_{-1}^{-1}+\daleth_1^1+\aleph_1^1\aleph_{-1}^{-1}\right)\right)-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0} \\ &-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\left(Q_1+Q_0\left(\daleth_{-1}^1+\daleth_1^{-1}+\aleph_1^{-1}\aleph_{-1}^1\right)\right)-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0} \\ &-\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\left(Q_1+Q_0\left(\daleth_{-1}^1+\daleth_1^1-\aleph_1^1\aleph_{-1}^1\right)\right)+\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0} \\ &-\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\left(Q_1+Q_0\left(\daleth_{-1}^{-1}+\daleth_1^{-1}-\aleph_1^{-1}\aleph_{-1}^{-1}\right)\right)+\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0} \\ &-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^o}{2}\left(\aleph_1^1-\aleph_{-1}^{-1}\right)+\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^o}{2}\left(\aleph_{-1}^1-\aleph_1^{-1}\right); \end{aligned}$$

$$\begin{aligned} \mathcal{O}\left(\frac{(z-a_j^o)^{-1/2}\,\mathrm{e}^{\mathrm{i}\widetilde{n}\Omega_j^o}}{\widetilde{n}G_{a_j}^o(z)}\right): &-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\left(Q_1\left(\aleph_1^1-\aleph_{-1}^{-1}\right)+Q_0\left(\beth_1^1-\beth_{-1}^{-1}+\aleph_1^1\daleth_{-1}^{-1}-\aleph_{-1}^{-1}\daleth_1^1\right)\right) \\ &-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\left(Q_1\left(\aleph_{-1}^1-\aleph_1^{-1}\right)+Q_0\left(\beth_{-1}^1-\beth_1^{-1}+\aleph_{-1}^1\daleth_1^{-1}-\aleph_1^{-1}\daleth_{-1}^1\right)\right) \\ &-\frac{\mathrm{i}(s_1-t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\left(Q_1\left(\aleph_{-1}^1+\aleph_1^1\right)+Q_0\left(\beth_{-1}^1+\beth_1^1+\aleph_1^1\daleth_{-1}^1+\aleph_{-1}^1\daleth_1^1\right)\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathrm{i}(s_1-t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\Big(Q_1\big(\aleph_{-1}^{-1}+\aleph_1^{-1}\big)+Q_0\big(\beth_{-1}^{-1}+\beth_1^{-1}+\aleph_1^{-1}\daleth_{-1}^{-1}+\aleph_{-1}^{-1}\daleth_1^{-1}\big)\Big)\\
&-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0}\big(\aleph_1^1-\aleph_{-1}^{-1}\big)-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o}{2}\big(\daleth_{-1}^{-1}+\daleth_1^1+\aleph_1^1\aleph_{-1}^{-1}\big)\\
&-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0}\big(\aleph_{-1}^1-\aleph_1^{-1}\big)+\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o}{2}\big(\daleth_{-1}^1+\daleth_1^{-1}+\aleph_1^{-1}\aleph_{-1}^1\big)\\
&+\frac{\mathrm{i}(s_1-t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0}\big(\aleph_{-1}^1+\aleph_1^1\big)-\frac{\mathrm{i}(s_1-t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0}\big(\aleph_1^{-1}+\aleph_{-1}^{-1}\big);\\
\mathcal{O}\left(\frac{\mathrm{e}^{\mathrm{i}\widetilde{n}\Omega_j^o}}{\widetilde{n}G_{a_j}^o(z)}\right):\,&-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\Big(Q_0\big(\gimel_{-1}^{-1}+\gimel_1^1+\daleth_1^1\daleth_{-1}^{-1}+\aleph_{-1}^{-1}\beth_1^1+\aleph_1^1\beth_{-1}^{-1}\big)+Q_1\big(\daleth_{-1}^{-1}+\daleth_1^1+\aleph_1^1\aleph_{-1}^{-1}\big)\\
&+\frac{1}{2}Q_2\Big)-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\Big(Q_0\big(\gimel_{-1}^1+\gimel_1^{-1}+\daleth_1^{-1}\daleth_{-1}^1+\aleph_1^{-1}\beth_{-1}^1+\aleph_{-1}^1\beth_1^{-1}\big)+\frac{1}{2}Q_2\\
&+Q_1\big(\daleth_{-1}^1+\daleth_1^{-1}+\aleph_1^{-1}\aleph_{-1}^1\big)\Big)-\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\Big(Q_0\big(\gimel_{-1}^1+\gimel_1^1+\daleth_1^1\daleth_{-1}^1-\aleph_1^1\beth_{-1}^1-\aleph_{-1}^1\beth_1^1\big)\\
&+\frac{1}{2}Q_2+Q_1\big(\daleth_{-1}^1+\daleth_1^1-\aleph_1^1\aleph_{-1}^1\big)\Big)-\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^o}{4(\gamma^o)^2}\Big(Q_0\big(\gimel_{-1}^{-1}+\gimel_1^{-1}+\daleth_1^{-1}\daleth_{-1}^{-1}-\aleph_1^{-1}\beth_{-1}^{-1}\\
&-\aleph_{-1}^{-1}\beth_1^{-1}\big)+\frac{1}{2}Q_2+Q_1\big(\daleth_{-1}^{-1}+\daleth_1^{-1}-\aleph_1^{-1}\aleph_{-1}^{-1}\big)\Big)-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0}\big(\daleth_{-1}^{-1}+\daleth_1^1+\aleph_1^1\aleph_{-1}^{-1}\\
&-Q_1(Q_0)^{-1}\big)-\frac{(s_1+t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0}\big(\daleth_{-1}^1+\daleth_1^{-1}+\aleph_1^{-1}\aleph_{-1}^1-Q_1(Q_0)^{-1}\big)-\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^o}{2}\\
&\times\big(\beth_1^1-\beth_{-1}^{-1}+\aleph_1^1\daleth_{-1}^{-1}-\aleph_{-1}^{-1}\daleth_1^1\big)+\frac{\mathrm{i}(s_1+t_1)\varkappa_1^o\varkappa_2^o}{2}\big(\beth_{-1}^1-\beth_1^{-1}+\aleph_{-1}^1\daleth_1^{-1}-\aleph_1^{-1}\daleth_{-1}^1\big)\\
&+\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0}\big(\daleth_{-1}^1+\daleth_1^1-\aleph_1^1\aleph_{-1}^1-Q_1(Q_0)^{-1}\big)+\frac{(s_1-t_1)\varkappa_1^o\varkappa_2^o(\gamma^o)^2}{4Q_0}\big(\daleth_{-1}^{-1}+\daleth_1^{-1}\\
&-\aleph_1^{-1}\aleph_{-1}^{-1}-Q_1(Q_0)^{-1}\big).
\end{aligned}
$$

Repeating the above analysis, *mutatis mutandis*, for the (1 2)- and (2 1)-elements, substituting $\aleph_1^{-1}=\aleph_1^1$, $\aleph_{-1}^{-1}=\aleph_{-1}^1$, $\daleth_1^{-1}=\daleth_1^1$, $\daleth_{-1}^{-1}=\daleth_{-1}^1$, $\beth_1^{-1}=\beth_1^1$, $\beth_{-1}^{-1}=\beth_{-1}^1$, $\gimel_1^{-1}=\gimel_1^1$, and $\gimel_{-1}^{-1}=\gimel_{-1}^1$ into the above (and resulting) 'coefficient equations', and simplifying, one shows that: (i) the coefficients of the terms that are $\mathcal{O}((z-a_j^o)^{-p/2}((n+\frac{1}{2})G_{a_j}^o(z))^{-1}\exp(\mathrm{i}(n+\frac{1}{2})\Omega_j^o))$, $p=1,3$, are equal to zero; and (ii) recalling from Lemma 4.7 that, for $z\in\mathbb{U}_{\delta_{a_j}}^o\setminus(-\infty,a_j^o)$, $j=1,\ldots,N$, $G_{a_j}^o(z)=_{z\to a_j^o}\widehat{\alpha}_0+\widehat{\alpha}_1(z-a_j^o)+\widehat{\alpha}_2(z-a_j^o)^2+\mathcal{O}((z-a_j^o)^3)$, where $\widehat{\alpha}_0=\widehat{\alpha}_0^o(a_j^o):=\frac{4}{3}f(a_j^o)$, $\widehat{\alpha}_1=\widehat{\alpha}_1^o(a_j^o):=\frac{4}{3}f'(a_j^o)$, and $\widehat{\alpha}_2=\widehat{\alpha}_2^o(a_j^o):=\frac{2}{7}f''(a_j^o)$, with $f(a_j^o)$, $f'(a_j^o)$, and $f''(a_j^o)$ given in Lemma 4.7, substituting the expansion for $G_{a_j}^o(z)$ (as $z\to a_j^o$, $j=1,\ldots,N$) into the remaining non-zero coefficient equations, collecting coefficients of like powers of $(z-a_j^o)^{-p}$, $p=0,1,2$, and continuing, analytically, the resulting (rational) expressions to $\partial\mathbb{U}_{\delta_{a_j}}^o$, $j=1,\ldots,N$, one arrives at, after a lengthy algebraic calculation and reinserting explicit $a_j^o$, $j=1,\ldots,N$, dependencies,

$$
\begin{aligned}
w_+^{\Sigma_\circlearrowright^o}(z)\underset{\substack{n\to\infty\\ z\in\partial\mathbb{U}_{\delta_{a_j}}^o}}{=}&\frac{1}{n+\frac{1}{2}}\left(\frac{\mathcal{A}^o(a_j^o)}{\widehat{\alpha}_0^o(a_j^o)(z-a_j^o)^2}+\frac{(\mathcal{B}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)-\mathcal{A}^o(a_j^o)\widehat{\alpha}_1^o(a_j^o))}{(\widehat{\alpha}_0^o(a_j^o))^2(z-a_j^o)}\right.\\
&\left.+\frac{\left(\mathcal{A}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)\left(\left(\frac{\widehat{\alpha}_1^o(a_j^o)}{\widehat{\alpha}_0^o(a_j^o)}\right)^2-\frac{\widehat{\alpha}_2^o(a_j^o)}{\widehat{\alpha}_0^o(a_j^o)}\right)-\mathcal{B}^o(a_j^o)\widehat{\alpha}_1^o(a_j^o)+\mathcal{C}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)\right)}{(\widehat{\alpha}_0^o(a_j^o))^2}\right)\\
&+\mathcal{O}\left(\frac{\sum_{k=1}^{\infty}f_k^o(n)(z-a_j^o)^k}{n+\frac{1}{2}}\right)+\mathcal{O}\left(\frac{\overset{o}{\mathfrak{M}}{}^{\infty}(z)f_{a_j}^o(n)(\overset{o}{\mathfrak{M}}{}^{\infty}(z))^{-1}}{(n+\frac{1}{2})^2(z-a_j^o)^3(G_{a_j}^o(z))^2}\right),
\end{aligned}
\tag{5.3}
$$

where $\mathcal{A}^o(a_j^o)$, $\mathcal{B}^o(a_j^o)$, $j=1,\ldots,N$, are defined in Theorem 2.3.1, Equations (2.25), (2.27), (2.28), (2.35)–

(2.45), (2.49), (2.56), and (2.57),

$$
\frac{\mathcal{C}^o(a_j^o)}{\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o}}:=\begin{pmatrix}
\boxed{\begin{gathered}
-\varkappa_1^o(a_j^o)\varkappa_2^o(a_j^o)\Big(s_1\Big\{\tfrac{Q_0^o(a_j^o)}{(\gamma^o(0))^2}\Big[\gimel_{-1}^1(a_j^o)\\
+\gimel_1^1(a_j^o)+\daleth_1^1(a_j^o)\daleth_{-1}^1(a_j^o)\Big]+\tfrac{Q_1^o(a_j^o)}{(\gamma^o(0))^2}\\
\times\Big[\daleth_1^1(a_j^o)+\daleth_{-1}^1(a_j^o)\Big]+\tfrac{1}{2}\tfrac{Q_2^o(a_j^o)}{(\gamma^o(0))^2}\\
+\tfrac{(\gamma^o(0))^2}{Q_0^o(a_j^o)}\aleph_1^1(a_j^o)\aleph_{-1}^1(a_j^o)\Big\}\\
+t_1\Big\{\tfrac{Q_0^o(a_j^o)}{(\gamma^o(0))^2}\Big[\aleph_{-1}^1(a_j^o)\beth_1^1(a_j^o)\\
+\aleph_1^1(a_j^o)\beth_{-1}^1(a_j^o)\Big]+\tfrac{(\gamma^o(0))^2}{Q_0^o(a_j^o)}\\
\times\Big[\daleth_{-1}^1(a_j^o)+\daleth_1^1(a_j^o)-\tfrac{Q_1^o(a_j^o)}{Q_0^o(a_j^o)}\Big]\\
+\tfrac{Q_1^o(a_j^o)}{(\gamma^o(0))^2}\aleph_1^1(a_j^o)\aleph_{-1}^1(a_j^o)\Big\}\\
+\mathrm{i}(s_1+t_1)\Big\{\beth_1^1(a_j^o)+\aleph_1^1(a_j^o)\daleth_{-1}^1(a_j^o)\\
-\beth_{-1}^1(a_j^o)-\aleph_{-1}^1(a_j^o)\daleth_1^1(a_j^o)\Big\}\Big)
\end{gathered}} &
\boxed{\begin{gathered}
(\varkappa_1^o(a_j^o))^2\Big(\mathrm{i}s_1\Big\{\tfrac{Q_0^o(a_j^o)}{(\gamma^o(0))^2}\Big[2\gimel_1^1(a_j^o)\\
+(\daleth_1^1(a_j^o))^2\Big]+\tfrac{2Q_1^o(a_j^o)}{(\gamma^o(0))^2}\daleth_1^1(a_j^o)\\
-\tfrac{(\gamma^o(0))^2}{Q_0^o(a_j^o)}(\aleph_1^1(a_j^o))^2+\tfrac{1}{2}\tfrac{Q_2^o(a_j^o)}{(\gamma^o(0))^2}\Big\}\\
+\mathrm{i}t_1\Big\{\tfrac{2Q_0^o(a_j^o)}{(\gamma^o(0))^2}\aleph_1^1(a_j^o)\beth_1^1(a_j^o)\\
+\tfrac{Q_1^o(a_j^o)}{(\gamma^o(0))^2}(\aleph_1^1(a_j^o))^2+\tfrac{(\gamma^o(0))^2}{Q_0^o(a_j^o)}\\
\times\Big[\tfrac{Q_1^o(a_j^o)}{Q_0^o(a_j^o)}-2\daleth_1^1(a_j^o)\Big]\Big\}-2(s_1-t_1)\\
\times\Big\{\beth_1^1(a_j^o)+\aleph_1^1(a_j^o)\daleth_1^1(a_j^o)\Big\}\Big)
\end{gathered}} \\
\boxed{\begin{gathered}
(\varkappa_2^o(a_j^o))^2\Big(\mathrm{i}s_1\Big\{\tfrac{Q_0^o(a_j^o)}{(\gamma^o(0))^2}\Big[2\gimel_{-1}^1(a_j^o)\\
+(\daleth_{-1}^1(a_j^o))^2\Big]+\tfrac{2Q_1^o(a_j^o)}{(\gamma^o(0))^2}\daleth_{-1}^1(a_j^o)\\
-\tfrac{(\gamma^o(0))^2}{Q_0^o(a_j^o)}(\aleph_{-1}^1(a_j^o))^2+\tfrac{1}{2}\tfrac{Q_2^o(a_j^o)}{(\gamma^o(0))^2}\Big\}\\
+\mathrm{i}t_1\Big\{\tfrac{2Q_0^o(a_j^o)}{(\gamma^o(0))^2}\aleph_{-1}^1(a_j^o)\beth_{-1}^1(a_j^o)\\
+\tfrac{Q_1^o(a_j^o)}{(\gamma^o(0))^2}(\aleph_{-1}^1(a_j^o))^2+\tfrac{(\gamma^o(0))^2}{Q_0^o(a_j^o)}\\
\times\Big[\tfrac{Q_1^o(a_j^o)}{Q_0^o(a_j^o)}-2\daleth_{-1}^1(a_j^o)\Big]\Big\}+2(s_1-t_1)\\
\times\Big\{\beth_{-1}^1(a_j^o)+\aleph_{-1}^1(a_j^o)\daleth_{-1}^1(a_j^o)\Big\}\Big)
\end{gathered}} &
\boxed{\begin{gathered}
\varkappa_1^o(a_j^o)\varkappa_2^o(a_j^o)\Big(s_1\Big\{\tfrac{Q_0^o(a_j^o)}{(\gamma^o(0))^2}\Big[\gimel_{-1}^1(a_j^o)\\
+\gimel_1^1(a_j^o)+\daleth_1^1(a_j^o)\daleth_{-1}^1(a_j^o)\Big]+\tfrac{Q_1^o(a_j^o)}{(\gamma^o(0))^2}\\
\times\Big[\daleth_1^1(a_j^o)+\daleth_{-1}^1(a_j^o)\Big]+\tfrac{1}{2}\tfrac{Q_2^o(a_j^o)}{(\gamma^o(0))^2}\\
+\tfrac{(\gamma^o(0))^2}{Q_0^o(a_j^o)}\aleph_1^1(a_j^o)\aleph_{-1}^1(a_j^o)\Big\}\\
+t_1\Big\{\tfrac{Q_0^o(a_j^o)}{(\gamma^o(0))^2}\Big[\aleph_{-1}^1(a_j^o)\beth_1^1(a_j^o)\\
+\aleph_1^1(a_j^o)\beth_{-1}^1(a_j^o)\Big]+\tfrac{(\gamma^o(0))^2}{Q_0^o(a_j^o)}\\
\times\Big[\daleth_{-1}^1(a_j^o)+\daleth_1^1(a_j^o)-\tfrac{Q_1^o(a_j^o)}{Q_0^o(a_j^o)}\Big]\\
+\tfrac{Q_1^o(a_j^o)}{(\gamma^o(0))^2}\aleph_1^1(a_j^o)\aleph_{-1}^1(a_j^o)\Big\}\\
+\mathrm{i}(s_1+t_1)\Big\{\beth_1^1(a_j^o)+\aleph_1^1(a_j^o)\daleth_{-1}^1(a_j^o)\\
-\beth_{-1}^1(a_j^o)-\aleph_{-1}^1(a_j^o)\daleth_1^1(a_j^o)\Big\}\Big)
\end{gathered}}
\end{pmatrix},
$$

(with $\mathrm{tr}(\mathcal{C}^o(a_j^o))=0$), and $(f_k^o(n))_{ij}=_{n\to\infty}\mathcal{O}(1)$, $k\in\mathbb{N}$, $i,j=1,2$. (The expression for $\mathcal{C}^o(a_j^o)$ is necessary for obtaining asymptotics **at** the end-points $\{a_j^o\}_{j=1}^N$, as well as for Remark 5.2 below.) Returning to the counter-clockwise-oriented integrals $\oint_{\partial\mathbb{U}_{\delta_{a_j}}^o}\frac{w_+^{\Sigma_\circlearrowright^o}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$, $z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^o$, it follows, via the Residue and Cauchy Theorems, that, for $j=1,\dots,N$,

$$
\oint_{\partial\mathbb{U}_{\delta_{a_j}}^o}\frac{w_+^{\Sigma_\circlearrowright^o}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\begin{cases}-\dfrac{\widehat{\mathcal{A}}^o(a_j^o)}{(n+\frac{1}{2})(z-a_j^o)^2}-\dfrac{\widehat{\mathcal{B}}^o(a_j^o)}{(n+\frac{1}{2})(z-a_j^o)}+\mathcal{O}\left(\dfrac{\widehat{f}^o(z;n)}{(n+\frac{1}{2})^2}\right), & z\in\mathbb{U}_{\delta_{a_j}}^{o,*},\\[2ex] -\dfrac{\widehat{\mathcal{A}}^o(a_j^o)}{(n+\frac{1}{2})(z-a_j^o)^2}-\dfrac{\widehat{\mathcal{B}}^o(a_j^o)}{(n+\frac{1}{2})(z-a_j^o)}+\dfrac{\mathcal{R}_{a_j^o}^0(z)}{n+\frac{1}{2}}+\mathcal{O}\left(\dfrac{\widehat{f}^o(z;n)}{(n+\frac{1}{2})^2}\right), & z\in\mathbb{U}_{\delta_{a_j}}^o,\end{cases}
$$

where $\mathbb{U}_{\delta_{a_j}}^{o,*}:=\mathbb{C}\setminus(\mathbb{U}_{\delta_{a_j}}^o\cup\partial\mathbb{U}_{\delta_{a_j}}^o)$, $\widehat{\mathcal{A}}^o(a_j^o):=(\widehat{\alpha}_0^o(a_j^o))^{-1}\mathcal{A}^o(a_j^o)$, $\widehat{\mathcal{B}}^o(a_j^o):=(\widehat{\alpha}_0^o(a_j^o))^{-2}(\mathcal{B}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)-\mathcal{A}^o(a_j^o)\widehat{\alpha}_1^o(a_j^o))$, $\mathcal{R}_{a_j^o}^o(z)$ is given in Theorem 2.3.1, Equations (2.73) and (2.74), and $\widehat{f}^o(z;n)$, where the $n$-dependence arises due to the $n$-dependence of the associated Riemann theta functions, denote some bounded (with respect to both $z$ and $n$), analytic (for $\mathbb{C}\setminus\widetilde{\Sigma}_p^o\ni z$), $\mathrm{GL}_2(\mathbb{C})$-valued functions for which $(\widehat{f}^o(z;n))_{kl}\underset{z\in\mathbb{C}\setminus\widetilde{\Sigma}_p^o}{=_{n\to\infty}}\mathcal{O}(1)$, $k,l=1,2$. Similarly, one shows that, for $j=1,\dots,N$,

$$
\oint_{\partial\mathbb{U}_{\delta_{a_j}}^o}s^{-1}w_+^{\Sigma_\circlearrowright^o}(s)\frac{\mathrm{d}s}{2\pi\mathrm{i}}\underset{n\to\infty}{=}\frac{(\mathcal{B}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)-\mathcal{A}^o(a_j^o)(\widehat{\alpha}_1^o(a_j^o)+(a_j^o)^{-1}\widehat{\alpha}_0^o(a_j^o)))}{(n+\frac{1}{2})(\widehat{\alpha}_0^o(a_j^o))^2a_j^o}+\mathcal{O}\left(\frac{f_j(n)}{(n+\frac{1}{2})^2}\right),
$$

where $(f_j(n))_{kl} =_{n\to\infty} \mathcal{O}(1)$, $k,l=1,2$. Repeating the above analysis for the remaining end-points of the support of the 'odd' equilibrium measure, that is, $\{b_0^o,\dots,b_N^o,a_{N+1}^o\}$, one arrives at the result stated in the Lemma. $\square$

**Remark 5.2.** A brisk perusing of the asymptotic (as $n\to\infty$) result for $\mathcal{R}^o(z)$ stated in Lemma 5.3 seems to imply that, at first glance, there are second-order poles at $\{b_{j-1}^o,a_j^o\}_{j=1}^{N+1}$; however, this is not the case. As the proof of Lemma 5.3 demonstrates (cf. the analysis leading up to Equations (5.3)), Laurent series expansions about $\{b_{j-1}^o,a_j^o\}_{j=1}^{N+1}$ show that, as $n\to\infty$, all expansions are, indeed, analytic; in particular: (i) for $z\in\mathbb{U}_{\delta_{a_j}}^o$, $j=1,\dots,N+1$ (all contour integrals are counter-clockwise oriented),

$$\oint_{\partial\mathbb{U}_{\delta_{a_j}}^o}\frac{w_+^{\Sigma_\cup^o}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{\substack{n\to\infty\\ z\in\mathbb{U}_{\delta_{a_j}}^o}}{=} \frac{\left(\mathcal{A}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)\left(\left(\frac{\widehat{\alpha}_1^o(a_j^o)}{\widehat{\alpha}_0^o(a_j^o)}\right)^2-\frac{\widehat{\alpha}_2^o(a_j^o)}{\widehat{\alpha}_0^o(a_j^o)}\right)-\mathcal{B}^o(a_j^o)\widehat{\alpha}_1^o(a_j^o)+\mathcal{C}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)\right)}{(n+\frac{1}{2})(\widehat{\alpha}_0^o(a_j^o))^2}$$
$$+\frac{1}{(n+\frac{1}{2})}\sum_{k=1}^{\infty}f_k^{a_j^o}(n)(z-a_j^o)^k+\mathcal{O}\left(\frac{\widehat{f^o}(z;n)}{(n+\frac{1}{2})^2}\right),$$

where, for $j=1,\dots,N$, $\mathcal{A}^o(a_j^o)$, $\mathcal{B}^o(a_j^o)$, $\mathcal{C}^o(a_j^o)$, $\widehat{\alpha}_0^o(a_j^o)$, $\widehat{\alpha}_1^o(a_j^o)$, and $\widehat{\alpha}_2^o(a_j^o)$ are given in (the proof of) Lemma 5.3, $\mathcal{A}^o(a_{N+1}^o)$, $\mathcal{B}^o(a_{N+1}^o)$ are given in Theorem 2.3.1, Equations (2.25), (2.27), (2.28), (2.31), (2.32), (2.37)–(2.45), (2.47), (2.52), and (2.53), $\widehat{\alpha}_0^o(a_{N+1}^o):=\frac{4}{3}f(a_{N+1}^o)$, $\widehat{\alpha}_1^o(a_{N+1}^o):=\frac{4}{5}f'(a_{N+1}^o)$, and $\widehat{\alpha}_2^o(a_{N+1}^o):=\frac{2}{7}f''(a_{N+1}^o)$, with $f(a_{N+1}^o)$, $f'(a_{N+1}^o)$, and $f''(a_{N+1}^o)$ given in Lemma 4.7, $\mathcal{C}^o(a_{N+1}^o)$ is given by the same expression as $\mathcal{C}^o(a_j^o)$ above subject to the modifications $\Omega_j^o\to 0$, $a_j^o\to a_{N+1}^o$, $Q_0^o(a_j^o)\to Q_0^o(a_{N+1}^o)$, $Q_1^o(a_j^o)\to Q_1^o(a_{N+1}^o)$, with $Q_0^o(a_{N+1}^o)$, $Q_1^o(a_{N+1}^o)$ given in Theorem 2.3.1, Equations (2.31) and (2.32), and $Q_2^o(a_j^o)\to Q_2^o(a_{N+1}^o)$, where

$$Q_2^o(a_{N+1}^o)=-\frac{1}{2}Q_0^o(a_{N+1}^o)\left(\sum_{k=1}^{N}\left(\frac{1}{(a_{N+1}^o-b_k^o)^2}-\frac{1}{(a_{N+1}^o-a_k^o)^2}\right)+\frac{1}{(a_{N+1}^o-b_0^o)^2}\right)$$
$$+\frac{1}{4}Q_0^o(a_{N+1}^o)\left(\sum_{k=1}^{N}\left(\frac{1}{a_{N+1}^o-b_k^o}-\frac{1}{a_{N+1}^o-a_k^o}\right)+\frac{1}{a_{N+1}^o-b_0^o}\right)^2,$$

$\widehat{f^o}(z;n)$ is characterised completely at the end of the proof of Lemma 5.3, $(f_k^{a_j^o}(n))_{l_1l_2}=_{n\to\infty}\mathcal{O}(1)$, $k\in\mathbb{N}$, $l_1,l_2=1,2$, and

$$\oint_{\partial\mathbb{U}_{\delta_{a_{N+1}}}^o}s^{-1}w_+^{\Sigma_\cup^o}(s)\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \frac{(\mathcal{B}^o(a_{N+1}^o)\widehat{\alpha}_0^o(a_{N+1}^o)-\mathcal{A}^o(a_{N+1}^o)(\widehat{\alpha}_1^o(a_{N+1}^o)+(a_{N+1}^o)^{-1}\widehat{\alpha}_0^o(a_{N+1}^o)))}{(n+\frac{1}{2})(\widehat{\alpha}_0^o(a_{N+1}^o))^2a_{N+1}^o}$$
$$+\mathcal{O}\left(\frac{f(n)}{(n+\frac{1}{2})^2}\right),$$

where $(f(n))_{kl}=_{n\to\infty}\mathcal{O}(1)$, $k,l=1,2$: and (ii) for $z\in\mathbb{U}_{\delta_{b_j}}^o$, $j=0,\dots,N$,

$$\oint_{\partial\mathbb{U}_{\delta_{b_j}}^o}\frac{w_+^{\Sigma_\cup^o}(s)}{s-z}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{\substack{n\to\infty\\ z\in\mathbb{U}_{\delta_{b_j}}^o}}{=} \frac{\left(\mathcal{A}^o(b_j^o)\widehat{\alpha}_0^o(b_j^o)\left(\left(\frac{\widehat{\alpha}_1^o(b_j^o)}{\widehat{\alpha}_0^o(b_j^o)}\right)^2-\frac{\widehat{\alpha}_2^o(b_j^o)}{\widehat{\alpha}_0^o(b_j^o)}\right)-\mathcal{B}^o(b_j^o)\widehat{\alpha}_1^o(b_j^o)+\mathcal{C}^o(b_j^o)\widehat{\alpha}_0^o(b_j^o)\right)}{(n+\frac{1}{2})(\widehat{\alpha}_0^o(b_j^o))^2}$$
$$+\frac{1}{(n+\frac{1}{2})}\sum_{k=1}^{\infty}f_k^{b_j^o}(n)(z-b_j^o)^k+\mathcal{O}\left(\frac{\widehat{f^o}(z;n)}{(n+\frac{1}{2})^2}\right),$$

where, for $j=1,\dots,N+1$, $\mathcal{A}^o(b_{j-1}^o)$, $\mathcal{B}^o(b_{j-1}^o)$ are given in Theorem 2.3.1, Equations (2.24), (2.26), (2.28), (2.29), (2.30), (2.33), (2.34), (2.37)–(2.45), (2.46), (2.48), (2.50), (2.51), (2.54), and (2.55), $\widehat{\alpha}_0^o(b_{j-1}^o):=\frac{4}{3}f(b_{j-1}^o)$, $\widehat{\alpha}_1^o(b_{j-1}^o):=\frac{4}{5}f'(b_{j-1}^o)$, and $\widehat{\alpha}_2^o(b_{j-1}^o):=\frac{2}{7}f''(b_{j-1}^o)$, with $f(b_{j-1}^o)$, $f'(b_{j-1}^o)$, and $f''(b_{j-1}^o)$ given in Lemma 4.6,

for $j=1,\dots,N$,

$$\frac{\mathcal{C}^{o}(b_j^o)}{\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\Omega_j^o}} := \begin{pmatrix}
\begin{gathered}
\varkappa_1^o(b_j^o)\varkappa_2^o(b_j^o)\Big(s_1\Big\{-\tfrac{Q_0^o(b_j^o)}{(\gamma^o(0))^2}\aleph_1^1(b_j^o) \\
\times\aleph_{-1}^1(b_j^o)-\tfrac{(Q_1^o(b_j^o)\gamma^o(0))^2}{(Q_0^o(b_j^o))^3}+\tfrac{1}{2}\tfrac{Q_2^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2} \\
+\tfrac{Q_1^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2}\Big[\daleth_1^1(b_j^o)+\daleth_{-1}^1(b_j^o)\Big]-\tfrac{(\gamma^o(0))^2}{Q_0^o(b_j^o)} \\
\times\Big[\gimel_1^1(b_j^o)+\gimel_{-1}^1(b_j^o)+\daleth_1^1(b_j^o)\daleth_{-1}^1(b_j^o)\Big]\Big\} \\
+t_1\Big\{-\tfrac{Q_1^o(b_j^o)}{(\gamma^o(0))^2}-\tfrac{Q_0^o(b_j^o)}{(\gamma^o(0))^2}\Big[\daleth_{-1}^1(b_j^o)+\daleth_1^1(b_j^o)\Big] \\
+\tfrac{Q_1^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2}\aleph_1^1(b_j^o)\aleph_{-1}^1(b_j^o)-\tfrac{(\gamma^o(0))^2}{Q_0^o(b_j^o)} \\
\times\Big[\aleph_1^1(b_j^o)\beth_{-1}^1(b_j^o)+\aleph_{-1}^1(b_j^o)\beth_1^1(b_j^o)\Big]\Big\} \\
+\mathrm{i}(s_1+t_1)\Big[\beth_{-1}^1(b_j^o)-\aleph_1^1(b_j^o)\daleth_{-1}^1(b_j^o) \\
+\aleph_{-1}^1(b_j^o)\daleth_1^1(b_j^o)-\beth_1^1(b_j^o)\Big]\Big)
\end{gathered}
&
\begin{gathered}
(\varkappa_1^o(b_j^o))^2\Big(\mathrm{i}s_1\Big\{\tfrac{Q_0^o(b_j^o)}{(\gamma^o(0))^2}(\aleph_1^1(b_j^o))^2 \\
-\tfrac{(Q_1^o(b_j^o)\gamma^o(0))^2}{(Q_0^o(b_j^o))^3}+\tfrac{1}{2}\tfrac{Q_2^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2} \\
+\tfrac{2Q_1^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2}\daleth_1^1(b_j^o)-\tfrac{(\gamma^o(0))^2}{Q_0^o(b_j^o)} \\
\times\Big[2\gimel_1^1(b_j^o)+(\daleth_1^1(b_j^o))^2\Big]\Big\}+\mathrm{i}t_1 \\
\times\Big\{\tfrac{2Q_0^o(b_j^o)}{(\gamma^o(0))^2}\daleth_1^1(b_j^o)+\tfrac{Q_1^o(b_j^o)}{(\gamma^o(0))^2} \\
+\tfrac{Q_1^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2}(\aleph_1^1(b_j^o))^2-\tfrac{2(\gamma^o(0))^2}{Q_0^o(b_j^o)} \\
\times\aleph_1^1(b_j^o)\beth_1^1(b_j^o)\Big\}+2(s_1-t_1) \\
\times\Big\{\beth_1^1(b_j^o)+\aleph_1^1(b_j^o)\daleth_1^1(b_j^o)\Big\}\Big)
\end{gathered}
\\
\begin{gathered}
(\varkappa_2^o(b_j^o))^2\Big(\mathrm{i}s_1\Big\{\tfrac{Q_0^o(b_j^o)}{(\gamma^o(0))^2}(\aleph_{-1}^1(b_j^o))^2 \\
-\tfrac{(Q_1^o(b_j^o)\gamma^o(0))^2}{(Q_0^o(b_j^o))^3}+\tfrac{1}{2}\tfrac{Q_2^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2} \\
+\tfrac{2Q_1^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2}\daleth_{-1}^1(b_j^o)-\tfrac{(\gamma^o(0))^2}{Q_0^o(b_j^o)} \\
\times\Big[2\gimel_{-1}^1(b_j^o)+(\daleth_{-1}^1(b_j^o))^2\Big]\Big\}+\mathrm{i}t_1 \\
\times\Big\{\tfrac{2Q_0^o(b_j^o)}{(\gamma^o(0))^2}\daleth_{-1}^1(b_j^o)+\tfrac{Q_1^o(b_j^o)}{(\gamma^o(0))^2} \\
+\tfrac{Q_1^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2}(\aleph_{-1}^1(b_j^o))^2-\tfrac{2(\gamma^o(0))^2}{Q_0^o(b_j^o)} \\
\times\aleph_{-1}^1(b_j^o)\beth_{-1}^1(b_j^o)\Big\}-2(s_1-t_1) \\
\times\Big\{\beth_{-1}^1(b_j^o)+\aleph_{-1}^1(b_j^o)\daleth_{-1}^1(b_j^o)\Big\}\Big)
\end{gathered}
&
\begin{gathered}
-\varkappa_1^o(b_j^o)\varkappa_2^o(b_j^o)\Big(s_1\Big\{-\tfrac{Q_0^o(b_j^o)}{(\gamma^o(0))^2}\aleph_1^1(b_j^o) \\
\times\aleph_{-1}^1(b_j^o)-\tfrac{(Q_1^o(b_j^o)\gamma^o(0))^2}{(Q_0^o(b_j^o))^3}+\tfrac{1}{2}\tfrac{Q_2^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2} \\
+\tfrac{Q_1^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2}\Big[\daleth_1^1(b_j^o)+\daleth_{-1}^1(b_j^o)\Big]-\tfrac{(\gamma^o(0))^2}{Q_0^o(b_j^o)} \\
\times\Big[\gimel_1^1(b_j^o)+\gimel_{-1}^1(b_j^o)+\daleth_1^1(b_j^o)\daleth_{-1}^1(b_j^o)\Big]\Big\} \\
+t_1\Big\{-\tfrac{Q_1^o(b_j^o)}{(\gamma^o(0))^2}-\tfrac{Q_0^o(b_j^o)}{(\gamma^o(0))^2}\Big[\daleth_{-1}^1(b_j^o)+\daleth_1^1(b_j^o)\Big] \\
+\tfrac{Q_1^o(b_j^o)(\gamma^o(0))^2}{(Q_0^o(b_j^o))^2}\aleph_1^1(b_j^o)\aleph_{-1}^1(b_j^o)-\tfrac{(\gamma^o(0))^2}{Q_0^o(b_j^o)} \\
\times\Big[\aleph_1^1(b_j^o)\beth_{-1}^1(b_j^o)+\aleph_{-1}^1(b_j^o)\beth_1^1(b_j^o)\Big]\Big\} \\
+\mathrm{i}(s_1+t_1)\Big[\beth_{-1}^1(b_j^o)-\aleph_1^1(b_j^o)\daleth_{-1}^1(b_j^o) \\
+\aleph_{-1}^1(b_j^o)\daleth_1^1(b_j^o)-\beth_1^1(b_j^o)\Big]\Big)
\end{gathered}
\end{pmatrix}$$

(with $\operatorname{tr}(\mathcal{C}^{o}(b_j^o))=0$), where $Q_0^o(b_j^o)$, $Q_1^o(b_j^o)$ are given in Theorem 2.3.1, Equations (2.33) and (2.34),

$$\begin{aligned}
Q_2^o(b_j^o) = &-\frac{1}{2}Q_0^o(b_j^o)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{(b_j^o-b_k^o)^2}-\frac{1}{(b_j^o-a_k^o)^2}\right)+\frac{1}{(b_j^o-b_0^o)^2}-\frac{1}{(b_j^o-a_{N+1}^o)^2}-\frac{1}{(b_j^o-a_j^o)^2}\right) \\
&+\frac{1}{4}Q_0^o(b_j^o)\left(\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{1}{b_j^o-b_k^o}-\frac{1}{b_j^o-a_k^o}\right)+\frac{1}{b_j^o-b_0^o}-\frac{1}{b_j^o-a_{N+1}^o}-\frac{1}{b_j^o-a_j^o}\right)^2,
\end{aligned}$$

$\mathcal{C}^{o}(b_0^o)$ is given by the same expression as $\mathcal{C}^{o}(b_j^o)$ above subject to the modifications $\Omega_j^o\to 0$, $b_j^o\to b_0^o$, $Q_0^o(b_j^o)\to Q_0^o(b_0^o)$, $Q_1^o(b_j^o)\to Q_1^o(b_0^o)$, with $Q_0^o(b_0^o)$, $Q_1^o(b_0^o)$ given in Theorem 2.3.1, Equations (2.29) and (2.30), and $Q_2^o(b_j^o)\to Q_2^o(b_0^o)$, where

$$\begin{aligned}
Q_2^o(b_0^o) = &-\frac{1}{2}Q_0^o(b_0^o)\left(\sum_{k=1}^{N}\left(\frac{1}{(b_0^o-b_k^o)^2}-\frac{1}{(b_0^o-a_k^o)^2}\right)-\frac{1}{(b_0^o-a_{N+1}^o)^2}\right) \\
&+\frac{1}{4}Q_0^o(b_0^o)\left(\sum_{k=1}^{N}\left(\frac{1}{b_0^o-b_k^o}-\frac{1}{b_0^o-a_k^o}\right)-\frac{1}{b_0^o-a_{N+1}^o}\right)^2,
\end{aligned}$$

$(f_k^{b_{j-1}^o}(n))_{l_1l_2} =_{n\to\infty} \mathcal{O}(1)$, $j=1,\dots,N+1$, $k\in\mathbb{N}$, $l_1,l_2=1,2$, and, for $j=1,\dots,N+1$,

$$\oint_{\partial\mathbb{U}^o_{\delta_{b_{j-1}}}} s^{-1}w_+^{\Sigma_\circlearrowright^o}(s)\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \underset{n\to\infty}{=} \frac{(\mathcal{B}^o(b_{j-1}^o)\widehat{\alpha}_0^o(b_{j-1}^o)-\mathcal{A}^o(b_{j-1}^o)(\widehat{\alpha}_1^o(b_{j-1}^o)+(b_{j-1}^o)^{-1}\widehat{\alpha}_0^o(b_{j-1}^o)))}{(n+\frac{1}{2})(\widehat{\alpha}_0^o(b_{j-1}^o))^2 b_{j-1}^o}$$

$$+\,\mathcal{O}\left(\frac{f_j(n)}{(n+\frac{1}{2})^2}\right),$$

where $(f_j(n))_{kl} =_{n\to\infty} \mathcal{O}(1)$, $k,l=1,2$. ■

Re-tracing the finite sequence of RHP transformations (all of which are invertible) and definitions, namely, $\mathcal{R}^o(z)$ (Lemmae 5.3 and 4.8) and $\mathcal{S}^o_p(z)$ (Lemma 4.8) $\to \mathcal{X}^o(z)$ (Lemmae 4.6 and 4.7) $\to \overset{o}{m}{}^{\infty}(z)$ (Lemma 4.5) $\to \overset{o}{\mathcal{M}}{}^{\sharp}(z)$ (Lemma 4.2) $\to \overset{o}{\mathcal{M}}{}^{\flat}(z)$ (Proposition 4.1) $\to \overset{o}{\mathcal{M}}(z)$ (Lemma 4.1) $\to \overset{o}{\mathrm{Y}}(z)$ (Lemma 3.4), the asymptotic (as $n\to\infty$) solution of the original **RHP2**, that is, $(\overset{o}{\mathrm{Y}}(z), \mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_+, \mathbb{R})$, in the various bounded and unbounded regions (Figure 7), is given by:

(1) for $z\in\Upsilon^o_1$,

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_o}{2}\,\mathrm{ad}(\sigma_3)}\mathcal{R}^o(z)\overset{o}{m}{}^{\infty}(z)\mathbb{E}^{\sigma_3}\mathrm{e}^{n(g^o(z)-\mathfrak{Q}^+_{\mathcal{A}})\sigma_3},$$

and, for $z\in\Upsilon^o_2$,

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_o}{2}\,\mathrm{ad}(\sigma_3)}\mathcal{R}^o(z)\overset{o}{m}{}^{\infty}(z)\mathbb{E}^{-\sigma_3}\mathrm{e}^{n(g^o(z)-\mathfrak{Q}^-_{\mathcal{A}})\sigma_3},$$

where $g^o(z)$ and $\mathfrak{Q}^{\pm}_{\mathcal{A}}$, $\ell_o$, $\mathbb{E}$, $\overset{o}{m}{}^{\infty}(z)$, and $\mathcal{R}^o(z)$ are given in Lemma 3.4, Lemma 3.6, Proposition 4.1, Lemma 4.5, and Lemma 5.3, respectively;

(2) for $z\in\Upsilon^o_3$,

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_o}{2}\,\mathrm{ad}(\sigma_3)}\mathcal{R}^o(z)\overset{o}{m}{}^{\infty}(z)\left(\mathrm{I}+\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\,\sigma_-\right)\mathbb{E}^{\sigma_3}\mathrm{e}^{n(g^o(z)-\mathfrak{Q}^+_{\mathcal{A}})\sigma_3},$$

where $\psi^o_V(z)$ is given in Lemma 3.5, and, for $z\in\Upsilon^o_4$,

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_o}{2}\,\mathrm{ad}(\sigma_3)}\mathcal{R}^o(z)\overset{o}{m}{}^{\infty}(z)\left(\mathrm{I}-\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\,\sigma_-\right)\mathbb{E}^{-\sigma_3}\mathrm{e}^{n(g^o(z)-\mathfrak{Q}^-_{\mathcal{A}})\sigma_3};$$

(3) for $z\in\Omega^{o,1}_{b_{j-1}}\cup\Omega^{o,1}_{a_j}$, $j=1,\dots,N+1$,

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_o}{2}\,\mathrm{ad}(\sigma_3)}\mathcal{R}^o(z)\mathcal{X}^o(z)\mathbb{E}^{\sigma_3}\mathrm{e}^{n(g^o(z)-\mathfrak{Q}^+_{\mathcal{A}})\sigma_3},$$

and, for $z\in\Omega^{o,4}_{b_{j-1}}\cup\Omega^{o,4}_{a_j}$, $j=1,\dots,N+1$,

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_o}{2}\,\mathrm{ad}(\sigma_3)}\mathcal{R}^o(z)\mathcal{X}^o(z)\mathbb{E}^{-\sigma_3}\mathrm{e}^{n(g^o(z)-\mathfrak{Q}^-_{\mathcal{A}})\sigma_3},$$

where, for $z\in\mathbb{U}^o_{\delta_{b_{j-1}}}$ $(\supset\Omega^{o,1}_{b_{j-1}}\cup\Omega^{o,4}_{b_{j-1}})$, $\mathcal{X}^o(z)$ is given by Lemma 4.6, and, for $z\in\mathbb{U}^o_{\delta_{a_j}}$ $(\supset\Omega^{o,1}_{a_j}\cup\Omega^{o,4}_{a_j})$, $\mathcal{X}^o(z)$ is given by Lemma 4.7; and

(4) for $z\in\Omega^{o,2}_{b_{j-1}}\cup\Omega^{o,2}_{a_j}$, $j=1,\dots,N+1$,

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_o}{2}\,\mathrm{ad}(\sigma_3)}\mathcal{R}^o(z)\mathcal{X}^o(z)\left(\mathrm{I}+\mathrm{e}^{-4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\,\sigma_-\right)\mathbb{E}^{\sigma_3}\mathrm{e}^{n(g^o(z)-\mathfrak{Q}^+_{\mathcal{A}})\sigma_3},$$

and, for $z\in\Omega^{o,3}_{b_{j-1}}\cup\Omega^{o,3}_{a_j}$, $j=1,\dots,N+1$,

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_o}{2}\,\mathrm{ad}(\sigma_3)}\mathcal{R}^o(z)\mathcal{X}^o(z)\left(\mathrm{I}-\mathrm{e}^{4(n+\frac{1}{2})\pi\mathrm{i}\int_z^{a^o_{N+1}}\psi^o_V(s)\,\mathrm{d}s}\,\sigma_-\right)\mathbb{E}^{-\sigma_3}\mathrm{e}^{n(g^o(z)-\mathfrak{Q}^-_{\mathcal{A}})\sigma_3}.$$

Multiplying the respective matrices in items (1)–(4) above and collecting (1 1)- and (1 2)-elements, one arrives at, finally, the asymptotic (as $n\to\infty$) results for $z\boldsymbol{\pi}_{2n+1}(z)$ and $\int_{\mathbb{R}}\frac{(s\boldsymbol{\pi}_{2n+1}(s))\exp(-n\widetilde{V}(s))}{s(s-z)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}$ (in the entire complex plane) stated in Theorem 2.3.1.

In order to obtain asymptotics (as $n\to\infty$) for $\xi^{(2n+1)}_{-n-1}$ $(=\|\boldsymbol{\pi}_{2n+1}(\cdot)\|^{-1}_{\mathcal{L}}=(H^{(-2n)}_{2n+1}/H^{(-2n-2)}_{2n+2})^{1/2})$ and $\phi_{2n+1}(z)$ $(=\xi^{(2n+1)}_{-n-1}\boldsymbol{\pi}_{2n+1}(z))$ stated in Theorem 2.3.2, small-$z$ asymptotics for $\overset{o}{\mathrm{Y}}(z)$ are necessary.

**Remark 5.3.** Since a tedious algebraic exercise shows that $\mathbb{C}_- \ni z \to 0$ asymptotics of $\overset{o}{m}^{\infty}(z)$ are obtained by multiplying $\mathbb{C}_+ \ni z \to 0$ asymptotics of $\overset{o}{m}^{\infty}(z)$ on the right by $\exp(\mathrm{i}(n+\frac{1}{2})\Omega_j^o\sigma_3)$ and using the relation $\mathbb{E}^{-\sigma_3}\exp(\mathrm{i}(n+\frac{1}{2})\Omega_j^o\sigma_3) = \mathbb{E}^{\sigma_3}$, only the asymptotic expansion as $\mathbb{C}_+ \ni z \to 0$ of $\overset{o}{m}^{\infty}(z)$ is presented in Proposition 5.3 below. ■

**Proposition 5.3.** *Let $\mathcal{R}^o\colon \mathbb{C} \setminus \widetilde{\Sigma}_p^o \to \mathrm{SL}_2(\mathbb{C})$ be the solution of the* RHP $(\mathcal{R}^o(z), v^o_{\mathcal{R}}(z), \widetilde{\Sigma}^o_p)$ *formulated in Proposition* 5.2 *with $n \to \infty$ asymptotics given in Lemma* 5.3*. Then,*

$$\mathcal{R}^o(z) \underset{z\to 0}{=} \mathrm{I} + \mathcal{R}_1^{o,0}(n)z + \mathcal{R}_2^{o,0}(n)z^2 + \mathcal{O}(z^3),$$

*where, for $k=2,3$,*

$$\mathcal{R}^{o,0}_{k-1}(n) := \int_{\Sigma^o_{\circlearrowright}} s^{-k} w_+^{\Sigma^o_{\circlearrowright}}(s)\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} = -\sum_{j=1}^{N+1}\sum_{q\in\{b^o_{j-1},a^o_j\}} \mathrm{Res}\left(z^{-k} w_+^{\Sigma^o_{\circlearrowright}}(z); q\right),$$

*with, in particular,*

$$\begin{aligned}\mathcal{R}^{o,0}_{k-1}(n) \underset{n\to\infty}{=} \frac{1}{(n+\frac{1}{2})}\sum_{j=1}^{N+1}\Bigg(&\frac{(\mathcal{A}^o(b^o_{j-1})(\widehat{\alpha}^o_1(b^o_{j-1}) + k(b^o_{j-1})^{-1}\widehat{\alpha}^o_0(b^o_{j-1})) - \mathcal{B}^o(b^o_{j-1})\widehat{\alpha}^o_0(b^o_{j-1}))}{(b^o_{j-1})^k(\widehat{\alpha}^o_0(b^o_{j-1}))^2}\\ &+ \frac{(\mathcal{A}^o(a^o_j)(\widehat{\alpha}^o_1(a^o_j) + k(a^o_j)^{-1}\widehat{\alpha}^o_0(a^o_j)) - \mathcal{B}^o(a^o_j)\widehat{\alpha}^o_0(a^o_j))}{(a^o_j)^k(\widehat{\alpha}^o_0(a^o_j))^2}\Bigg) + \mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right),\end{aligned}$$

*and all parameters are defined in Lemma* 5.3*.*

*Let $\overset{o}{m}^{\infty}\colon \mathbb{C} \setminus J_o^{\infty} \to \mathrm{SL}_2(\mathbb{C})$ solve the* RHP $(\overset{o}{m}^{\infty}(z), J_o^{\infty}, \overset{o}{v}^{\infty}(z))$ *formulated in Lemma* 4.3 *with (unique) solution given by Lemma* 4.5*. For $\varepsilon_1, \varepsilon_2 = \pm 1$, set*

$$\begin{aligned}\theta^o_0(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o) &:= \boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o_+(0) - \tfrac{1}{2\pi}(n+\tfrac{1}{2})\boldsymbol{\Omega}^o + \varepsilon_2\boldsymbol{d}_o),\\ \alpha^o_0(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o) &:= 2\pi\mathrm{i}\varepsilon_1\sum_{m\in\mathbb{Z}^N}(m,\widehat{\boldsymbol{\alpha}}^o_0)\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}^o_+(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)+\pi\mathrm{i}(m,\tau^o m)},\end{aligned}$$

*where $\widehat{\boldsymbol{\alpha}}^o_0 = (\widehat{\alpha}^o_{0,1}, \widehat{\alpha}^o_{0,2}, \dots, \widehat{\alpha}^o_{0,N})$, with $\widehat{\alpha}^o_{0,j} := (-1)^{\mathcal{N}_+}(\prod_{i=1}^{N+1}|b^o_{i-1}a^o_i|)^{-1/2}c^o_{jN}$, $j=1,\dots,N$, where $\mathcal{N}_+ \in \{0,\dots,N+1\}$ is the number of bands to the right of $z=0$, and*

$$\beta^o_0(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o) := 2\pi\sum_{m\in\mathbb{Z}^N}\left(\mathrm{i}\varepsilon_1(m,\widehat{\boldsymbol{\beta}}^o_0) - \pi(m,\widehat{\boldsymbol{\alpha}}^o_0)^2\right)\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}^o_+(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)+\pi\mathrm{i}(m,\tau^o m)},$$

*where $\widehat{\boldsymbol{\beta}}^o_0 = (\widehat{\beta}^o_{0,1}, \widehat{\beta}^o_{0,2}, \dots, \widehat{\beta}^o_{0,N})$, with $\widehat{\beta}^o_{0,j} := \frac{1}{2}(-1)^{\mathcal{N}_+}(\prod_{i=1}^{N+1}|b^o_{i-1}a^o_i|)^{-1/2}(c^o_{jN-1} + \frac{1}{2}c^o_{jN}\sum_{k=1}^{N+1}((a^o_k)^{-1} + (b^o_{k-1})^{-1}))$, $j=1,\dots,N$, where $c^o_{jN}, c^o_{jN-1}$, $j=1,\dots,N$, are obtained from Equations* (O1) *and* (O2)*. Then,*

$$\overset{o}{m}^{\infty}(z) \underset{z\to 0}{=} \mathbb{E}^{-\sigma_3} + \overset{o}{m}{}^0_1 z + \overset{o}{m}{}^0_2 z^2 + \mathcal{O}(z^3),$$

*where*

$$\begin{aligned}(\overset{o}{m}{}^0_1)_{11} &= -\frac{\boldsymbol{\theta}^o(\boldsymbol{u}^o_+(0)+\boldsymbol{d}_o)\mathbb{E}^{-1}}{\boldsymbol{\theta}^o(\boldsymbol{u}^o_+(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)}\left(\frac{\theta^o_0(1,1,\boldsymbol{\Omega}^o)\alpha^o_0(1,1,\vec{\boldsymbol{0}}) - \alpha^o_0(1,1,\boldsymbol{\Omega}^o)\theta^o_0(1,1,\vec{\boldsymbol{0}})}{(\theta^o_0(1,1,\vec{\boldsymbol{0}}))^2}\right),\\ (\overset{o}{m}{}^0_1)_{12} &= \frac{1}{4\mathrm{i}}\left(\sum_{k=1}^{N+1}\left(\frac{1}{b^o_{k-1}} - \frac{1}{a^o_k}\right)\right)\frac{\boldsymbol{\theta}^o(\boldsymbol{u}^o_+(0)+\boldsymbol{d}_o)\theta^o_0(-1,1,\boldsymbol{\Omega}^o)\mathbb{E}^{-1}}{\boldsymbol{\theta}^o(\boldsymbol{u}^o_+(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\boldsymbol{d}_o)\theta^o_0(-1,1,\vec{\boldsymbol{0}})},\\ (\overset{o}{m}{}^0_1)_{21} &= -\frac{1}{4\mathrm{i}}\left(\sum_{k=1}^{N+1}\left(\frac{1}{b^o_{k-1}} - \frac{1}{a^o_k}\right)\right)\frac{\boldsymbol{\theta}^o(\boldsymbol{u}^o_+(0)+\boldsymbol{d}_o)\theta^o_0(1,-1,\boldsymbol{\Omega}^o)\mathbb{E}}{\boldsymbol{\theta}^o(-\boldsymbol{u}^o_+(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o-\boldsymbol{d}_o)\theta^o_0(1,-1,\vec{\boldsymbol{0}})},\\ (\overset{o}{m}{}^0_1)_{22} &= -\left(\frac{\theta^o_0(-1,-1,\boldsymbol{\Omega}^o)\alpha^o_0(-1,-1,\vec{\boldsymbol{0}}) - \alpha^o_0(-1,-1,\boldsymbol{\Omega}^o)\theta^o_0(-1,-1,\vec{\boldsymbol{0}})}{(\theta^o_0(-1,-1,\vec{\boldsymbol{0}}))^2}\right)\end{aligned}$$

$$\times \frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})},$$

$$(\overset{o}{m}{}_{2}^{0})_{11} = \Big(\theta_{0}^{o}(1,1,\boldsymbol{\Omega}^{o})\Big(-\beta_{0}^{0}(1,1,\vec{\boldsymbol{0}})\theta_{0}^{o}(1,1,\vec{\boldsymbol{0}})+(\alpha_{0}^{o}(1,1,\vec{\boldsymbol{0}}))^{2}\Big)-\alpha_{0}^{o}(1,1,\boldsymbol{\Omega}^{o})\alpha_{0}^{o}(1,1,\vec{\boldsymbol{0}})\theta_{0}^{o}(1,1,\vec{\boldsymbol{0}})$$
$$+\beta_{0}^{o}(1,1,\boldsymbol{\Omega}^{o})(\theta_{0}^{o}(1,1,\vec{\boldsymbol{0}}))^{2}\Big)\frac{(\theta_{0}^{o}(1,1,\vec{\boldsymbol{0}}))^{-3}\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}+\frac{1}{32}\left(\sum_{k=1}^{N+1}\left(\frac{1}{b_{k-1}^{o}}-\frac{1}{a_{k}^{o}}\right)\right)^{2}\mathbb{E}^{-1},$$

$$(\overset{o}{m}{}_{2}^{0})_{12} = -\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}\left(\left(\frac{\theta_{0}^{o}(-1,1,\boldsymbol{\Omega}^{o})\alpha_{0}^{o}(-1,1,\vec{\boldsymbol{0}})-\alpha_{0}^{o}(-1,1,\boldsymbol{\Omega}^{o})\theta_{0}^{o}(-1,1,\vec{\boldsymbol{0}})}{(\theta_{0}^{o}(-1,1,\vec{\boldsymbol{0}}))^{2}}\right)\right.$$
$$\left.\times\frac{1}{4\mathrm{i}}\left(\sum_{k=1}^{N+1}\left(\frac{1}{b_{k-1}^{o}}-\frac{1}{a_{k}^{o}}\right)\right)-\frac{1}{8\mathrm{i}}\left(\sum_{k=1}^{N+1}\left(\frac{1}{(b_{k-1}^{o})^{2}}-\frac{1}{(a_{k}^{o})^{2}}\right)\right)\frac{\theta_{0}^{o}(-1,1,\boldsymbol{\Omega}^{o})}{\theta_{0}^{o}(-1,1,\vec{\boldsymbol{0}})}\right),$$

$$(\overset{o}{m}{}_{2}^{0})_{21} = \frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\left(\left(\frac{\theta_{0}^{o}(1,-1,\boldsymbol{\Omega}^{o})\alpha_{0}^{o}(1,-1,\vec{\boldsymbol{0}})-\alpha_{0}^{o}(1,-1,\boldsymbol{\Omega}^{o})\theta_{0}^{o}(1,-1,\vec{\boldsymbol{0}})}{(\theta_{0}^{o}(1,-1,\vec{\boldsymbol{0}}))^{2}}\right)\right.$$
$$\left.\times\frac{1}{4\mathrm{i}}\left(\sum_{k=1}^{N+1}\left(\frac{1}{b_{k-1}^{o}}-\frac{1}{a_{k}^{o}}\right)\right)-\frac{1}{8\mathrm{i}}\left(\sum_{k=1}^{N+1}\left(\frac{1}{(b_{k-1}^{o})^{2}}-\frac{1}{(a_{k}^{o})^{2}}\right)\right)\frac{\theta_{0}^{o}(1,-1,\boldsymbol{\Omega}^{o})}{\theta_{0}^{o}(1,-1,\vec{\boldsymbol{0}})}\right),$$

$$(\overset{o}{m}{}_{2}^{0})_{22} = \Big(\theta_{0}^{o}(-1,-1,\boldsymbol{\Omega}^{o})\Big(-\beta_{0}^{o}(-1,-1,\vec{\boldsymbol{0}})\theta_{0}^{o}(-1,-1,\vec{\boldsymbol{0}})+(\alpha_{0}^{o}(-1,-1,\vec{\boldsymbol{0}}))^{2}\Big)-\alpha_{0}^{o}(-1,-1,\boldsymbol{\Omega}^{o})$$
$$\times\alpha_{0}^{o}(-1,-1,\vec{\boldsymbol{0}})\theta_{0}^{o}(-1,-1,\vec{\boldsymbol{0}})+\beta_{0}^{o}(-1,-1,\boldsymbol{\Omega}^{o})(\theta_{0}^{o}(-1,-1,\vec{\boldsymbol{0}}))^{2}\Big)(\theta_{0}^{o}(-1,-1,\vec{\boldsymbol{0}}))^{-3}$$
$$\times\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}_{+}^{o}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}_{+}^{o}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}+\frac{1}{32}\left(\sum_{k=1}^{N+1}\left(\frac{1}{b_{k-1}^{o}}-\frac{1}{a_{k}^{o}}\right)\right)^{2}\mathbb{E},$$

*with* $(\star)_{ij}$, $i,j=1,2$, *denoting the* $(i\,j)$*-element of* $\star$, $\mathbb{E}$ *defined in Proposition* 4.1, *and* $\vec{\boldsymbol{0}}:=(0,0,\dots,0)^{\mathrm{T}}$ $(\in\mathbb{R}^{N})$.

*Let* $\overset{o}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_{2}(\mathbb{C})$ *be the solution of* **RHP2**. *Then,*

$$\overset{o}{\mathrm{Y}}(z)z^{n\sigma_{3}} \underset{z\to0}{=} \mathrm{I}+z\mathrm{Y}_{1}^{o,0}+z^{2}\mathrm{Y}_{2}^{o,0}+\mathcal{O}(z^{3}),$$

*where*

$$(\mathrm{Y}_{1}^{o,0})_{11} = -(2n+1)\int_{J_{o}}s^{-1}\psi_{V}^{o}(s)\,\mathrm{d}s+(\overset{o}{m}{}_{1}^{0})_{11}\mathbb{E}+(\mathcal{R}_{1}^{o,0}(n))_{11},$$
$$(\mathrm{Y}_{1}^{o,0})_{12} = \mathrm{e}^{n\ell_{o}}\Big((\overset{o}{m}{}_{1}^{0})_{12}\mathbb{E}^{-1}+(\mathcal{R}_{1}^{o,0}(n))_{12}\Big),$$
$$(\mathrm{Y}_{1}^{o,0})_{21} = \mathrm{e}^{-n\ell_{o}}\Big((\overset{o}{m}{}_{1}^{0})_{21}\mathbb{E}+(\mathcal{R}_{1}^{o,0}(n))_{21}\Big),$$
$$(\mathrm{Y}_{1}^{o,0})_{22} = (2n+1)\int_{J_{o}}s^{-1}\psi_{V}^{o}(s)\,\mathrm{d}s+(\overset{o}{m}{}_{1}^{0})_{22}\mathbb{E}^{-1}+(\mathcal{R}_{1}^{o,0}(n))_{22},$$

$$(\mathrm{Y}_{2}^{o,0})_{11} = \tfrac{1}{2}(2n+1)^{2}\left(\int_{J_{o}}s^{-1}\psi_{V}^{o}(s)\,\mathrm{d}s\right)^{2}-\tfrac{1}{2}(2n+1)\int_{J_{o}}s^{-2}\psi_{V}^{o}(s)\,\mathrm{d}s-(2n+1)\Big((\overset{o}{m}{}_{1}^{0})_{11}\mathbb{E}+(\mathcal{R}_{1}^{o,0}(n))_{11}\Big)$$
$$\times\int_{J_{o}}s^{-1}\psi_{V}^{o}(s)\,\mathrm{d}s+(\overset{o}{m}{}_{2}^{0})_{11}\mathbb{E}+(\mathcal{R}_{2}^{o,0}(n))_{11}+\Big((\mathcal{R}_{1}^{o,0}(n))_{11}(\overset{o}{m}{}_{1}^{0})_{11}+(\mathcal{R}_{1}^{o,0}(n))_{12}(\overset{o}{m}{}_{1}^{0})_{21}\Big)\mathbb{E},$$

$$(\mathrm{Y}_{2}^{o,0})_{12} = \mathrm{e}^{n\ell_{o}}\Big((2n+1)\Big((\overset{o}{m}{}_{1}^{0})_{12}\mathbb{E}^{-1}+(\mathcal{R}_{1}^{o,0}(n))_{12}\Big)\int_{J_{o}}s^{-1}\psi_{V}^{o}(s)\,\mathrm{d}s+(\overset{o}{m}{}_{2}^{0})_{12}\mathbb{E}^{-1}+(\mathcal{R}_{2}^{o,0}(n))_{12}$$
$$+\Big((\mathcal{R}_{1}^{o,0}(n))_{11}(\overset{o}{m}{}_{1}^{0})_{12}+(\mathcal{R}_{1}^{o,0}(n))_{12}(\overset{o}{m}{}_{1}^{0})_{22}\Big)\mathbb{E}^{-1}\Big),$$

$$(\mathrm{Y}_{2}^{o,0})_{21} = \mathrm{e}^{-n\ell_{o}}\Big(-(2n+1)\Big((\overset{o}{m}{}_{1}^{0})_{21}\mathbb{E}+(\mathcal{R}_{1}^{o,0}(n))_{21}\Big)\int_{J_{o}}s^{-1}\psi_{V}^{o}(s)\,\mathrm{d}s+(\overset{o}{m}{}_{2}^{0})_{21}\mathbb{E}+(\mathcal{R}_{2}^{o,0}(n))_{21}$$
$$+\Big((\mathcal{R}_{1}^{o,0}(n))_{21}(\overset{o}{m}{}_{1}^{0})_{11}+(\mathcal{R}_{1}^{o,0}(n))_{22}(\overset{o}{m}{}_{1}^{0})_{21}\Big)\mathbb{E}\Big),$$

$$(\mathrm{Y}_{2}^{o,0})_{22} = \tfrac{1}{2}(2n+1)^{2}\left(\int_{J_{o}}s^{-1}\psi_{V}^{o}(s)\,\mathrm{d}s\right)^{2}+\tfrac{1}{2}(2n+1)\int_{J_{o}}s^{-2}\psi_{V}^{o}(s)\,\mathrm{d}s+(2n+1)\Big((\overset{o}{m}{}_{1}^{0})_{22}\mathbb{E}^{-1}+(\mathcal{R}_{1}^{o,0}(n))_{22}\Big)$$

$$\times \int_{J_o} s^{-1}\psi_V^o(s)\,\mathrm{d}s+(\overset{o}{m}{}_2^0)_{22}\mathbb{E}^{-1}+(\mathcal{R}_2^{o,0}(n))_{22}+\Big((\mathcal{R}_1^{o,0}(n))_{21}(\overset{o}{m}{}_1^0)_{12}+(\mathcal{R}_1^{o,0}(n))_{22}(\overset{o}{m}{}_1^0)_{22}\Big)\mathbb{E}^{-1}.$$

*Proof.* Let $\mathcal{R}^o\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^o\to\mathrm{SL}_2(\mathbb{C})$ be the solution of the RHP $(\mathcal{R}^o(z),v_{\mathcal{R}}^o(z),\widetilde{\Sigma}_p^o)$ formulated in Proposition 5.2 with $n\to\infty$ asymptotics given in Lemma 5.3. For $|z|\ll\min_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$, via the expansion $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, where $s\in\{b_{j-1}^o,a_j^o\}$, $j=1,\dots,N+1$, one obtains the asymptotics for $\mathcal{R}^o(z)$ stated in the Proposition.

Let $\overset{o}{m}{}^\infty\colon\mathbb{C}\setminus J_o^\infty\to\mathrm{SL}_2(\mathbb{C})$ solve the RHP $(\overset{o}{m}{}^\infty(z),J_o^\infty,\overset{o}{v}{}^\infty(z))$ formulated in Lemma 4.3 with (unique) solution given by Lemma 4.5. In order to obtain small-$z$ asymptotics of $\overset{o}{m}{}^\infty(z)$, one needs small-$z$ asymptotics of $(\gamma^o(z))^{\pm1}$ and $\frac{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o(z)+\varepsilon_2\boldsymbol{d}_o)}$, $\varepsilon_1,\varepsilon_2=\pm1$. Consider, say, and without loss of generality, $z\to0$ asymptotics for $z\in\mathbb{C}_+$ (designated $z\to0^+$), where, by definition, $\sqrt{\star(z)}:=+\sqrt{\star(z)}$: equivalently, one may consider $z\to0$ asymptotics for $z\in\mathbb{C}_-$ (designated $z\to0^-$); however, recalling that $\sqrt{\star(z)}\restriction_{\mathbb{C}_+}=-\sqrt{\star(z)}\restriction_{\mathbb{C}_-}$, one obtains (in either case, and via the sheet-interchange index) the same $z\to0$ asymptotics (for $\overset{o}{m}{}^\infty(z)$). Recall the expression for $\gamma^o(z)$ given in Lemma 4.4: for $|z|\ll\min_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$, via the expansions $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, and $\ln(s-z)=_{|z|\to0}\ln(s)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{s})^k$, where $s\in\{b_{j-1}^o,a_j^o\}$, $j=1,\dots,N+1$, one shows that, upon setting $\gamma_0^o:=\gamma^o(0)=(\prod_{k=1}^{N+1}b_{k-1}^o(a_k^o)^{-1})^{1/4}$ $(>0)$,

$$(\gamma_0^o)^{\mp1}(\gamma^o(z))^{\pm1}\underset{z\to0^+}{=}1+z\left(\pm\frac{1}{4}\sum_{k=1}^{N+1}\left(\frac{1}{a_k^o}-\frac{1}{b_{k-1}^o}\right)\right)+z^2\left(\pm\frac{1}{8}\sum_{k=1}^{N+1}\left(\frac{1}{(a_k^o)^2}-\frac{1}{(b_{k-1}^o)^2}\right)\right.$$
$$\left.+\frac{1}{32}\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^o}-\frac{1}{b_{k-1}^o}\right)\right)^2\right)+O(z^3),$$

whence

$$\frac{1}{2}((\gamma_0^o)^{-1}\gamma^o(z)+\gamma_0^o(\gamma^o(z))^{-1})\underset{z\to0^+}{=}1+z^2\left(\frac{1}{32}\left(\sum_{k=1}^{N+1}\left(\frac{1}{a_k^o}-\frac{1}{b_{k-1}^o}\right)\right)^2\right)+O(z^3),$$

and

$$\frac{1}{2\mathrm{i}}((\gamma_0^o)^{-1}\gamma^o(z)-\gamma_0^o(\gamma^o(z))^{-1})\underset{z\to0^+}{=}z\left(\frac{1}{4\mathrm{i}}\sum_{k=1}^{N+1}\left(\frac{1}{a_k^o}-\frac{1}{b_{k-1}^o}\right)\right)+z^2\left(\frac{1}{8\mathrm{i}}\sum_{k=1}^{N+1}\left(\frac{1}{(a_k^o)^2}-\frac{1}{(b_{k-1}^o)^2}\right)\right)$$
$$+O(z^3).$$

Recall from Lemma 4.5 that $\boldsymbol{u}^o(z):=\int_{a_{N+1}^o}^{z}\boldsymbol{\omega}^o$ $(\in\mathrm{Jac}(\mathcal{Y}_o)$, with $\mathcal{Y}_o:=\{(y,z);\,y^2=R_o(z)\})$, where $\boldsymbol{\omega}^o$, the associated normalised basis of holomorphic one-forms of $\mathcal{Y}_o$, is given by $\boldsymbol{\omega}^o=(\omega_1^o,\omega_2^o,\dots,\omega_N^o)$, with $\omega_j^o:=\sum_{k=1}^{N}c_{jk}^o(\prod_{i=1}^{N+1}(z-b_{i-1}^o)(z-a_i^o))^{-1/2}z^{N-k}\,\mathrm{d}z$, $j=1,\dots,N$, where $c_{jk}^o$, $j,k=1,\dots,N$, are obtained from Equations (O1) and (O2). Writing

$$\boldsymbol{u}^o(z)=\left(\int_{a_{N+1}^o}^{0^+}+\int_{0^+}^{z}\right)\boldsymbol{\omega}^o=\boldsymbol{u}_+^o(0)+\int_{0^+}^{z}\boldsymbol{\omega}^o,$$

where $\boldsymbol{u}_+^o(0):=\int_{a_{N+1}^o}^{0^+}\boldsymbol{\omega}^o$ (cf. Lemma 4.5), for $|z|\ll\min_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$, via the expansions $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^k}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}_0^+$, and $\ln(z-s)=_{|z|\to0}\ln(s)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{s})^k$, where $s\in\{b_{k-1}^o,a_k^o\}$, $k=1,\dots,N+1$, one shows that, for $j=1,\dots,N$,

$$\omega_j^o\underset{z\to0^+}{=}(-1)^{\mathcal{N}_+}\left(\prod_{i=1}^{N+1}|b_{i-1}^oa_i^o|\right)^{-1/2}\left(c_{jN}^o\,\mathrm{d}z+\left(c_{jN-1}^o+\frac{c_{jN}^o}{2}\sum_{k=1}^{N+1}\left(\frac{1}{b_{k-1}^o}+\frac{1}{a_k^o}\right)\right)z\,\mathrm{d}z+O(z^2\,\mathrm{d}z)\right),$$

where $\mathcal{N}_+\in\{0,\dots,N+1\}$ is the number of bands to the right of $z=0$, whence

$$\int_{0^+}^{z}\omega_j^o\underset{z\to0^+}{=}(-1)^{\mathcal{N}_+}\left(\prod_{i=1}^{N+1}|b_{i-1}^oa_i^o|\right)^{-1/2}\left(c_{jN}^oz+\frac{1}{2}\left(c_{jN-1}^o+\frac{c_{jN}^o}{2}\sum_{k=1}^{N+1}\left(\frac{1}{b_{k-1}^o}+\frac{1}{a_k^o}\right)\right)z^2+O(z^3)\right)$$

$$=:\widehat{\alpha}^{o}_{0,j}z+\widehat{\beta}^{o}_{0,j}z^{2}+O(z^{3}).$$

Defining $\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})$, $\alpha^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})$, and $\beta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})$, $\varepsilon_{1},\varepsilon_{2}=\pm1$, as in the Proposition, recalling that $\boldsymbol{\omega}^{o}=(\omega^{o}_{1},\omega^{o}_{2},\dots,\omega^{o}_{N})$, and that the associated $N\times N$ Riemann matrix of $\boldsymbol{\beta}^{o}$-periods, $\tau^{o}=(\tau^{o})_{i,j=1,\dots,N}:=(\oint_{\boldsymbol{\beta}^{o}_{j}}\omega^{o}_{i})_{i,j=1,\dots,N}$, is non-degenerate, symmetric, and $-\mathrm{i}\tau^{o}$ is positive definite, via the above asymptotic (as $z\to0^{+}$) expansion for $\int_{0^{+}}^{z}\omega^{o}_{j}$, $j=1,\dots,N$, one shows that

$$\frac{\boldsymbol{\theta}^{o}(\varepsilon_{1}\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_{2}\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\varepsilon_{1}\boldsymbol{u}^{o}(z)+\varepsilon_{2}\boldsymbol{d}_{o})}\underset{z\to0^{+}}{=}F^{o}_{0}+F^{o}_{1}z+F^{o}_{2}z^{2}+O(z^{3}),$$

where

$$\begin{aligned}F^{o}_{0}&:=\frac{\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})}{\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})},\\ F^{o}_{1}&:=\frac{\alpha^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})-\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})\alpha^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})}{(\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}))^{2}},\\ F^{o}_{2}&:=\Big(\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})\Big((\alpha^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}))^{2}-\beta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})\Big)-\alpha^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})\\ &\quad\times\alpha^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})+\beta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})(\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}))^{2}\Big)(\theta^{o}_{0}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}))^{-3},\end{aligned}$$

with $\vec{\boldsymbol{0}}:=(0,0,\dots,0)^{\mathrm{T}}$ $(\in\mathbb{R}^{N})$. Via the above asymptotic (as $z\to0^{+}$) expansions for $\frac{1}{2}((\gamma^{o}_{0})^{-1}\gamma^{o}(z)+\gamma^{o}_{0}(\gamma^{o}(z))^{-1})$, $\frac{1}{2\mathrm{i}}((\gamma^{o}_{0})^{-1}\gamma^{o}(z)-\gamma^{o}_{0}(\gamma^{o}(z))^{-1})$, and $\frac{\boldsymbol{\theta}^{o}(\varepsilon_{1}\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_{2}\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\varepsilon_{1}\boldsymbol{u}^{o}(z)+\varepsilon_{2}\boldsymbol{d}_{o})}$, one arrives at, upon recalling the expression for $\overset{o}{m}{}^{\infty}(z)$ given in Lemma 4.5, the asymptotic expansion for $\overset{o}{m}{}^{\infty}(z)$ stated in the Proposition.

Let $\overset{o}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_{2}(\mathbb{C})$ be the (unique) solution of **RHP2**, that is, $(\overset{o}{\mathrm{Y}}(z),\mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_{+},\mathbb{R})$. Recall, also, that, for $z\in\Upsilon^{o}_{1}$ (Figure 7),

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_{o}}{2}\operatorname{ad}(\sigma_{3})}\mathcal{R}^{o}(z)\overset{o}{m}{}^{\infty}(z)\mathbb{E}^{\sigma_{3}}\mathrm{e}^{n(g^{o}(z)-\mathfrak{Q}^{+}_{\mathscr{A}})\sigma_{3}},$$

and, for $z\in\Upsilon^{o}_{2}$ (Figure 7),

$$\overset{o}{\mathrm{Y}}(z)=\mathrm{e}^{\frac{n\ell_{o}}{2}\operatorname{ad}(\sigma_{3})}\mathcal{R}^{o}(z)\overset{o}{m}{}^{\infty}(z)\mathbb{E}^{-\sigma_{3}}\mathrm{e}^{n(g^{o}(z)-\mathfrak{Q}^{-}_{\mathscr{A}})\sigma_{3}}:$$

consider, say, and without loss of generality, small-$z$ asymptotics for $\overset{o}{\mathrm{Y}}(z)$ for $z\in\Upsilon^{o}_{1}$. Recalling from Lemma 3.4 that $g^{o}(z):=\int_{J_{o}}\ln((z-s)^{2+\frac{1}{n}}(zs)^{-1})\psi^{o}_{V}(s)\,\mathrm{d}s$, $z\in\mathbb{C}\setminus(-\infty,\max\{0,a^{o}_{N+1}\})$, for $|z|\ll\min_{j=1,\dots,N+1}\{|b^{o}_{j-1}-a^{o}_{j}|\}$, in particular, $|z/s|\ll1$ with $s\in J_{o}$, and noting that $\int_{J_{o}}\psi^{o}_{V}(s)\,\mathrm{d}s=1$ and $\int_{J_{o}}s^{-m}\psi^{o}_{V}(s)\,\mathrm{d}s<\infty$, $m\in\mathbb{N}$, via the expansions $\frac{1}{z-s}=-\sum_{k=0}^{l}\frac{z^{k}}{s^{k+1}}+\frac{z^{l+1}}{s^{l+1}(z-s)}$, $l\in\mathbb{Z}^{+}_{0}$, and $\ln(s-z)=_{|z|\to0}\ln(s)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{s})^{k}$, one shows that

$$\begin{aligned}g^{o}(z)\underset{\mathbb{C}_{\pm}\ni z\to0}{=}&-\ln(z)+\Big(1+\frac{1}{n}\Big)\int_{J_{o}}\ln(|s|)\psi^{o}_{V}(s)\,\mathrm{d}s-\mathrm{i}\pi\int_{J_{o}\cap\mathbb{R}_{-}}\psi^{o}_{V}(s)\,\mathrm{d}s\pm\mathrm{i}\pi\Big(2+\frac{1}{n}\Big)\int_{J_{o}\cap\mathbb{R}_{+}}\psi^{o}_{V}(s)\,\mathrm{d}s\\ &+z\bigg(-\Big(2+\frac{1}{n}\Big)\int_{J_{o}}s^{-1}\psi^{o}_{V}(s)\,\mathrm{d}s\bigg)+z^{2}\bigg(-\frac{1}{2}\Big(2+\frac{1}{n}\Big)\int_{J_{o}}s^{-2}\psi^{o}_{V}(s)\,\mathrm{d}s\bigg)+O(z^{3}),\end{aligned}$$

where $\int_{J_{o}\cap\mathbb{R}_{\pm}}\psi^{o}_{V}(s)\,\mathrm{d}s$ are given in Lemma 3.4. (Explicit expressions for $\int_{J_{o}}s^{-k}\psi^{o}_{V}(s)\,\mathrm{d}s$, $k=1,2$, are given in Remark 3.2.) Using the asymptotic (as $z\to0$) expansions for $g^{o}(z)$, $\mathcal{R}^{o}(z)$, and $\overset{o}{m}{}^{\infty}(z)$ derived above, upon recalling the formula for $\overset{o}{\mathrm{Y}}(z)$, one arrives at, after a matrix-multiplication argument, the asymptotic expansion for $\overset{o}{\mathrm{Y}}(z)z^{n\sigma_{3}}$ stated in the Proposition. ◻

**Proposition 5.4.** *Let $\overset{o}{\mathrm{Y}}\colon\mathbb{C}\setminus\mathbb{R}\to\mathrm{SL}_{2}(\mathbb{C})$ be the solution of* **RHP2** *with $z$ $(\in\mathbb{C}\setminus\mathbb{R})\to0$ asymptotics given in Proposition 5.3. Then,*

$$\xi^{(2n+1)}_{-n-1}=\frac{1}{\|\boldsymbol{\pi}_{2n+1}(\cdot)\|_{\mathcal{L}}}=\sqrt{\frac{H^{(-2n)}_{2n+1}}{H^{(-2n-2)}_{2n+2}}}=\bigg(\frac{1}{2\pi\mathrm{i}(\mathrm{Y}^{o,0}_{1})_{12}}\bigg)^{1/2}\qquad(>0),$$

*where* $(\mathrm{Y}_1^{o,0})_{12} = \mathrm{e}^{n\ell_o}\Big((\overset{o}{m}{}_1^0)_{12}\mathbb{E}^{-1} + (\mathcal{R}_1^{o,0}(n))_{12}\Big)$, *with* $(\overset{o}{m}{}_1^0)_{12}$ *and* $(\mathcal{R}_1^{o,0}(n))_{12}$ *given in Proposition* 5.3.

*Proof.* Recall from Lemma 2.2.2 that $z\boldsymbol{\pi}_{2n+1}(z) := (\overset{o}{\mathrm{Y}}(z))_{11}$ and $(\overset{o}{\mathrm{Y}}(z))_{12} = z\int_{\mathbb{R}} \frac{(s\boldsymbol{\pi}_{2n+1}(s))\mathrm{e}^{-n\widetilde{V}(s)}}{s(s-z)} \cdot \frac{\mathrm{d}s}{2\pi\mathrm{i}}$. Using (for $|z/s| \ll 1$) the expansion $\frac{1}{z-s} = -\sum_{k=0}^{l} \frac{z^k}{s^{k+1}} + \frac{z^{l+1}}{s^{l+1}(z-s)}$, $l \in \mathbb{Z}_0^+$, and recalling that $\langle \boldsymbol{\pi}_{2n+1}, z^j \rangle_{\mathcal{L}} = 0$, $j = -n, \dots, n$, and $\phi_{2n+1}(z) = \xi_{-n-1}^{(2n+1)}(z)\boldsymbol{\pi}_{2n+1}(z)$, one proceeds as follows:

$$
\begin{aligned}
\Big(\overset{o}{\mathrm{Y}}(z)\Big)_{12} &\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=} z\int_{\mathbb{R}} \frac{\boldsymbol{\pi}_{2n+1}(s)}{s}\left(1+\frac{z}{s}+\dots+\frac{z^{n-1}}{s^{n-1}}+\frac{z^n}{s^n}+\cdots\right)\mathrm{e}^{-n\widetilde{V}(s)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\
&\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=} z\int_{\mathbb{R}} \boldsymbol{\pi}_{2n+1}(s)\left(\frac{z^n}{s^{n+1}}\right)\mathrm{e}^{-n\widetilde{V}(s)}\,\frac{\mathrm{d}s}{2\pi\mathrm{i}} + O\Big(z^{n+2}\Big) \\
&\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=} \frac{z^{n+1}}{\xi_{-n-1}^{(2n+1)}}\int_{\mathbb{R}} \underbrace{\xi_{-n-1}^{(2n+1)}\boldsymbol{\pi}_{2n+1}(s)}_{=\phi_{2n+1}(s)}\,\frac{\mathrm{e}^{-n\widetilde{V}(s)}}{\xi_{-n-1}^{(2n+1)}}\underbrace{\left(\xi_n^{(2n+1)}s^n+\dots+\frac{\xi_{-n-1}^{(2n+1)}}{s^{n+1}}\right)}_{=\phi_{2n+1}(s)}\frac{\mathrm{d}s}{2\pi\mathrm{i}} \\
&\quad + O\Big(z^{n+2}\Big) \\
&\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=} \frac{z^{n+1}}{2\pi\mathrm{i}(\xi_{-n-1}^{(2n+1)})^2}\underbrace{\int_{\mathbb{R}} \phi_{2n+1}(s)\phi_{2n+1}(s)\mathrm{e}^{-n\widetilde{V}(s)}\,\mathrm{d}s}_{=1} + O\Big(z^{n+2}\Big) \Rightarrow \\
\Big(\overset{o}{\mathrm{Y}}(z)z^{n\sigma_3}\Big)_{12} &\underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=} z\left(\frac{1}{2\pi\mathrm{i}(\xi_{-n-1}^{(2n+1)})^2}\right) + O(z^2);
\end{aligned}
$$

but, noting from Proposition 5.3 that

$$
\Big(\overset{o}{\mathrm{Y}}(z)z^{n\sigma_3}\Big)_{12} \underset{\substack{z\to 0\\ z\in\mathbb{C}\setminus\mathbb{R}}}{=} z\Big(\mathrm{Y}_1^{o,0}\Big)_{12} + O(z^2),
$$

upon equating the above two asymptotic expansions for $(\overset{o}{\mathrm{Y}}(z)z^{n\sigma_3})_{12}$, one arrives at the result stated in the Proposition. $\square$

Using the results of Propositions 5.3 and 5.4, one obtains $n \to \infty$ asymptotics for $\xi_{-n-1}^{(2n+1)}$ and $\phi_{2n+1}(z)$ (in the entire complex plane) stated in Theorem 2.3.2.

Large-$z$ asymptotics for $\overset{o}{\mathrm{Y}}(z)$ are given in the Appendix (see Lemma A.1): these latter asymptotics are necessary for the results of [40].

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# Appendix: Large-$z$ Asymptotics for $\overset{o}{\mathrm{Y}}(z)$

Even though the results of Lemma A.1 below, namely, large-$z$ asymptotics (as $(\mathbb{C}\setminus\mathbb{R}\ni)\, z\to\infty$) of $\overset{o}{\mathrm{Y}}(z)$, are not necessary in order to prove Theorems 2.3.1 and 2.3.2, they are essential for the results of [40], related to asymptotics of the coefficients of the system of three- and five-term recurrence relations and the corresponding Laurent-Jacobi matrices (cf. Section 1). For the sake of completeness, therefore, and in order to eschew any duplication of the analysis of this paper, $(\mathbb{C}\setminus\mathbb{R}\ni)\, z\to\infty$ asymptotics for $\overset{o}{\mathrm{Y}}(z)$ are presented here.

**Lemma A.1.** *Let* $\mathcal{R}^o\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^o\to \mathrm{SL}_2(\mathbb{C})$ *be the solution of the* RHP $(\mathcal{R}^o(z), \upsilon_{\mathcal{R}}^o(z), \widetilde{\Sigma}_p^o)$ *formulated in Proposition* 5.2 *with* $n\to\infty$ *asymptotics given in Lemma* 5.3*. Then,*

$$\mathcal{R}^o(z) \underset{z\to\infty}{=} \mathrm{I}+\mathcal{R}_0^{o,\infty}(n)+\frac{1}{z}\mathcal{R}_1^{o,\infty}(n)+\frac{1}{z^2}\mathcal{R}_2^{o,\infty}(n)+\mathcal{O}(z^{-3}),$$

*where, for* $k=-1,0,1,$

$$\mathcal{R}_{k+1}^{o,\infty}(n):=-\int_{\Sigma_{\circlearrowright}^o} s^k w_+^{\Sigma_{\circlearrowright}^o}(s)\,\frac{\mathrm{d}s}{2\pi\mathrm{i}}=\sum_{j=1}^{N+1}\sum_{q\in\{b_{j-1}^o,a_j^o\}}\mathrm{Res}\Big(z^k w_+^{\Sigma_{\circlearrowright}^o}(z);q\Big),$$

*with, in particular,*

$$\begin{aligned}
\mathcal{R}_0^{o,\infty}(n) \underset{n\to\infty}{=}& \frac{1}{(n+\frac{1}{2})}\sum_{j=1}^{N+1}\left(\frac{(\mathcal{B}^o(b_{j-1}^o)\widehat{\alpha}_0^o(b_{j-1}^o)-\mathcal{A}^o(b_{j-1}^o)(\widehat{\alpha}_1^o(b_{j-1}^o)+(b_{j-1}^o)^{-1}\widehat{\alpha}_0^o(b_{j-1}^o)))}{b_{j-1}^o(\widehat{\alpha}_0^o(b_{j-1}^o))^2}\right.\\
&+\left.\frac{(\mathcal{B}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)-\mathcal{A}^o(a_j^o)(\widehat{\alpha}_1^o(a_j^o)+(a_j^o)^{-1}\widehat{\alpha}_0^o(a_j^o)))}{a_j^o(\widehat{\alpha}_0^o(a_j^o))^2}\right)+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right),\\
\mathcal{R}_1^{o,\infty}(n) \underset{n\to\infty}{=}& \frac{1}{(n+\frac{1}{2})}\sum_{j=1}^{N+1}\left(\frac{(\mathcal{B}^o(b_{j-1}^o)\widehat{\alpha}_0^o(b_{j-1}^o)-\mathcal{A}^o(b_{j-1}^o)\widehat{\alpha}_1^o(b_{j-1}^o))}{(\widehat{\alpha}_0^o(b_{j-1}^o))^2}\right.\\
&+\left.\frac{(\mathcal{B}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)-\mathcal{A}^o(a_j^o)\widehat{\alpha}_1^o(a_j^o))}{(\widehat{\alpha}_0^o(a_j^o))^2}\right)+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right),\\
\mathcal{R}_2^{o,\infty}(n) \underset{n\to\infty}{=}& \frac{1}{(n+\frac{1}{2})}\sum_{j=1}^{N+1}\left(\frac{(\mathcal{B}^o(b_{j-1}^o)\widehat{\alpha}_0^o(b_{j-1}^o)b_{j-1}^o-\mathcal{A}^o(b_{j-1}^o)(b_{j-1}^o\widehat{\alpha}_1^o(b_{j-1}^o)-\widehat{\alpha}_0^o(b_{j-1}^o)))}{(\widehat{\alpha}_0^o(b_{j-1}^o))^2}\right.\\
&+\left.\frac{(\mathcal{B}^o(a_j^o)\widehat{\alpha}_0^o(a_j^o)a_j^o-\mathcal{A}^o(a_j^o)(a_j^o\widehat{\alpha}_1^o(a_j^o)-\widehat{\alpha}_0^o(a_j^o)))}{(\widehat{\alpha}_0^o(a_j^o))^2}\right)+\mathcal{O}\left(\frac{1}{(n+\frac{1}{2})^2}\right),
\end{aligned}$$

*and all parameters are defined in Lemma* 5.3*.*

*Let* $\overset{o}{m}{}^{\infty}\colon \mathbb{C}\setminus J_o^\infty\to\mathrm{SL}_2(\mathbb{C})$ *solve the* RHP $(\overset{o}{m}{}^{\infty}(z), J_o^\infty, \overset{o}{\upsilon}{}^{\infty}(z))$ *formulated in Lemma* 4.3 *with (unique) solution given by Lemma* 4.5*. For* $\varepsilon_1,\varepsilon_2=\pm1$*, set*

$$\theta_\infty^o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o):=\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}_+^o(\infty)-\tfrac{1}{2\pi}(n+\tfrac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o),$$

*where*[17] $\boldsymbol{u}_+^o(\infty)=\int_{a_{N+1}^o}^{\infty^+}\boldsymbol{\omega}^o$,

$$\widetilde{\alpha}_\infty^o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o):=2\pi\mathrm{i}\varepsilon_1\sum_{m\in\mathbb{Z}^N}(m,\widehat{\boldsymbol{\alpha}}_\infty^o)\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}_+^o(\infty)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)+\pi\mathrm{i}(m,\tau^o m)},$$

*where* $\widehat{\boldsymbol{\alpha}}_\infty^o=(\widehat{\alpha}_{\infty,1}^o,\widehat{\alpha}_{\infty,2}^o,\dots,\widehat{\alpha}_{\infty,N}^o)$*, with* $\widehat{\alpha}_{\infty,j}^o:=c_{j1}^o$*,* $j=1,\dots,N,$

$$\beta_\infty^o(\varepsilon_1,\varepsilon_2,\boldsymbol{\Omega}^o):=2\pi\sum_{m\in\mathbb{Z}^N}\Big(\mathrm{i}\varepsilon_1(m,\widehat{\boldsymbol{\beta}}{}_\infty^o)+\pi(m,\widehat{\boldsymbol{\alpha}}_\infty^o)^2\Big)\mathrm{e}^{2\pi\mathrm{i}(m,\varepsilon_1\boldsymbol{u}_+^o(\infty)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)+\pi\mathrm{i}(m,\tau^o m)},$$

[17] For $\mathcal{P}:=(y,z)\in\{(z_1,z_2);\, z_1^2=R_o(z_2)\}$, $\mathcal{P}\to\infty^\pm\Leftrightarrow z\to\infty$, $y\sim\pm z^{N+1}$.

*where* $\widehat{\boldsymbol{\beta}}^{o}_{\infty}=(\widehat{\beta}^{o}_{\infty,1},\widehat{\beta}^{o}_{\infty,2},\dots,\widehat{\beta}^{o}_{\infty,N})$, *with* $\widehat{\beta}^{o}_{\infty,j}:=\frac{1}{2}(c^{o}_{j2}+\frac{1}{2}c^{o}_{j1}\sum_{i=1}^{N+1}(b^{o}_{i-1}+a^{o}_{i}))$, $j=1,\dots,N$, *where* $c^{o}_{j1},c^{o}_{j2}$, $j=1,\dots,N$, *are obtained from Equations* (O1) *and* (O2). *Set* $\gamma^{o}_{0}:=\gamma^{o}(0)=(\prod_{k=1}^{N+1}b^{o}_{k-1}(a^{o}_{k})^{-1})^{1/4}\ (>0)$. *Then,*

$$\overset{o}{m}{}^{\infty}(z)\underset{z\to\infty}{=}\overset{o}{m}{}^{\infty}_{0}+\frac{1}{z}\overset{o}{m}{}^{\infty}_{1}+\frac{1}{z^{2}}\overset{o}{m}{}^{\infty}_{2}+O(z^{-3}),$$

*where*

$$(\overset{o}{m}{}^{\infty}_{0})_{11}=\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{2}\right)\frac{\theta^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}})},$$

$$(\overset{o}{m}{}^{\infty}_{0})_{12}=\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{2\mathrm{i}}\right)\frac{\theta^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}})},$$

$$(\overset{o}{m}{}^{\infty}_{0})_{21}=-\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{2\mathrm{i}}\right)\frac{\theta^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}})},$$

$$(\overset{o}{m}{}^{\infty}_{0})_{22}=\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{2}\right)\frac{\theta^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})},$$

$$\begin{aligned}(\overset{o}{m}{}^{\infty}_{1})_{11}=&\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}\left(\left(\frac{\widetilde{\alpha}^{o}_{\infty}(1,1,\vec{\mathbf{0}})\theta^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})-\widetilde{\alpha}^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}})}{(\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}}))^{2}}\right)\right.\\&\left.\times\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{2}\right)-\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{8}\right)\left(\sum_{k=1}^{N+1}\left(a^{o}_{k}-b^{o}_{k-1}\right)\right)\frac{\theta^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}})}\right),\end{aligned}$$

$$\begin{aligned}(\overset{o}{m}{}^{\infty}_{1})_{12}=&\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}\left(\left(\frac{\widetilde{\alpha}^{o}_{\infty}(-1,1,\vec{\mathbf{0}})\theta^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})-\widetilde{\alpha}^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}})}{(\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}}))^{2}}\right)\right.\\&\left.\times\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{2\mathrm{i}}\right)-\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{8\mathrm{i}}\right)\left(\sum_{k=1}^{N+1}\left(a^{o}_{k}-b^{o}_{k-1}\right)\right)\frac{\theta^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}})}\right),\end{aligned}$$

$$\begin{aligned}(\overset{o}{m}{}^{\infty}_{1})_{21}=&-\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\left(\left(\frac{\widetilde{\alpha}^{o}_{\infty}(1,-1,\vec{\mathbf{0}})\theta^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})-\widetilde{\alpha}^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}})}{(\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}}))^{2}}\right)\right.\\&\left.\times\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{2\mathrm{i}}\right)-\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{8\mathrm{i}}\right)\left(\sum_{k=1}^{N+1}\left(a^{o}_{k}-b^{o}_{k-1}\right)\right)\frac{\theta^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}})}\right),\end{aligned}$$

$$\begin{aligned}(\overset{o}{m}{}^{\infty}_{1})_{22}=&\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\left(\left(\frac{\widetilde{\alpha}^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})\theta^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})-\widetilde{\alpha}^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})}{(\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}}))^{2}}\right)\right.\\&\left.\times\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{2}\right)-\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{8}\right)\left(\sum_{k=1}^{N+1}\left(a^{o}_{k}-b^{o}_{k-1}\right)\right)\frac{\theta^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})}\right),\end{aligned}$$

$$\begin{aligned}(\overset{o}{m}{}^{\infty}_{2})_{11}=&\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}\Big(\Big(\theta^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})\Big((\widetilde{\alpha}^{o}_{\infty}(1,1,\vec{\mathbf{0}}))^{2}+\beta^{o}_{\infty}(1,1,\vec{\mathbf{0}})\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}})\Big)\\&-\widetilde{\alpha}^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})\widetilde{\alpha}^{o}_{\infty}(1,1,\vec{\mathbf{0}})\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}})-\beta^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})(\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}}))^{2}\Big)\tfrac{1}{(\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}}))^{3}}\\&\times\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{2}\right)+\left(\frac{\widetilde{\alpha}^{o}_{\infty}(1,1,\vec{\mathbf{0}})\theta^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})-\widetilde{\alpha}^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}})}{(\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}}))^{2}}\right)\\&\times\left(\sum_{k=1}^{N+1}\left(b^{o}_{k-1}-a^{o}_{k}\right)\right)\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{8}\right)+\left(\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{16}\right)\sum_{k=1}^{N+1}\left((b^{o}_{k-1})^{2}-(a^{o}_{k})^{2}\right)\right.\\&\left.+\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{64}\right)\left(\sum_{k=1}^{N+1}\left(a^{o}_{k}-b^{o}_{k-1}\right)\right)^{2}\right)\frac{\theta^{o}_{\infty}(1,1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(1,1,\vec{\mathbf{0}})}\Bigg),\end{aligned}$$

$$(\overset{o}{m}{}^{\infty}_{2})_{12}=\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}^{-1}}{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\boldsymbol{d}_{o})}\Big(\Big(\theta^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})\Big((\widetilde{\alpha}^{o}_{\infty}(-1,1,\vec{\mathbf{0}}))^{2}+\beta^{o}_{\infty}(-1,1,\vec{\mathbf{0}})\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}})\Big)$$

$$
\begin{aligned}
&-\widetilde{\alpha}^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})\widetilde{\alpha}^{o}_{\infty}(-1,1,\vec{\mathbf{0}})\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}})-\beta^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})(\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}}))^{2}\Big)\tfrac{1}{(\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}}))^{3}}\\
&\times\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{2\mathrm{i}}\right)-\left(\frac{\widetilde{\alpha}^{o}_{\infty}(-1,1,\vec{\mathbf{0}})\theta^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})-\widetilde{\alpha}^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}})}{(\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}}))^{2}}\right)\\
&\times\left(\sum_{k=1}^{N+1}\Big(a^{o}_{k}-b^{o}_{k-1}\Big)\right)\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{8\mathrm{i}}\right)-\left(\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{16\mathrm{i}}\right)\sum_{k=1}^{N+1}\Big((a^{o}_{k})^{2}-(b^{o}_{k-1})^{2}\Big)\right.\\
&\left.-\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{64\mathrm{i}}\right)\left(\sum_{k=1}^{N+1}\Big(a^{o}_{k}-b^{o}_{k-1}\Big)\right)^{2}\right)\frac{\theta^{o}_{\infty}(-1,1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(-1,1,\vec{\mathbf{0}})}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
(\overset{o}{m}{}^{\infty}_{2})_{21}&=-\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\Big(\Big(\theta^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})\Big((\widetilde{\alpha}^{o}_{\infty}(1,-1,\vec{\mathbf{0}}))^{2}+\beta^{o}_{\infty}(1,-1,\vec{\mathbf{0}})\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}})\Big)\\
&-\widetilde{\alpha}^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})\widetilde{\alpha}^{o}_{\infty}(1,-1,\vec{\mathbf{0}})\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}})-\beta^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})(\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}}))^{2}\Big)\tfrac{1}{(\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}}))^{3}}\\
&\times\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{2\mathrm{i}}\right)-\left(\frac{\widetilde{\alpha}^{o}_{\infty}(1,-1,\vec{\mathbf{0}})\theta^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})-\widetilde{\alpha}^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}})}{(\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}}))^{2}}\right)\\
&\times\left(\sum_{k=1}^{N+1}\Big(a^{o}_{k}-b^{o}_{k-1}\Big)\right)\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{8\mathrm{i}}\right)-\left(\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{16\mathrm{i}}\right)\sum_{k=1}^{N+1}\Big((a^{o}_{k})^{2}-(b^{o}_{k-1})^{2}\Big)\right.\\
&\left.-\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{64\mathrm{i}}\right)\left(\sum_{k=1}^{N+1}\Big(a^{o}_{k}-b^{o}_{k-1}\Big)\right)^{2}\right)\frac{\theta^{o}_{\infty}(1,-1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(1,-1,\vec{\mathbf{0}})}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
(\overset{o}{m}{}^{\infty}_{2})_{22}&=\frac{\boldsymbol{\theta}^{o}(\boldsymbol{u}^{o}_{+}(0)+\boldsymbol{d}_{o})\mathbb{E}}{\boldsymbol{\theta}^{o}(-\boldsymbol{u}^{o}_{+}(0)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}-\boldsymbol{d}_{o})}\Big(\Big(\theta^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})\Big((\widetilde{\alpha}^{o}_{\infty}(-1,-1,\vec{\mathbf{0}}))^{2}+\beta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})\Big)\\
&-\widetilde{\alpha}^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})\widetilde{\alpha}^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})-\beta^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})(\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}}))^{2}\Big)\tfrac{1}{(\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}}))^{3}}\\
&\times\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{2}\right)+\left(\frac{\widetilde{\alpha}^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})\theta^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})-\widetilde{\alpha}^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})}{(\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}}))^{2}}\right)\\
&\times\left(\sum_{k=1}^{N+1}\Big(b^{o}_{k-1}-a^{o}_{k}\Big)\right)\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{8}\right)+\left(\left(\frac{\gamma^{o}_{0}-(\gamma^{o}_{0})^{-1}}{16}\right)\sum_{k=1}^{N+1}\Big((b^{o}_{k-1})^{2}-(a^{o}_{k})^{2}\Big)\right.\\
&\left.+\left(\frac{\gamma^{o}_{0}+(\gamma^{o}_{0})^{-1}}{64}\right)\left(\sum_{k=1}^{N+1}\Big(a^{o}_{k}-b^{o}_{k-1}\Big)\right)^{2}\right)\frac{\theta^{o}_{\infty}(-1,-1,\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(-1,-1,\vec{\mathbf{0}})}\Bigg),
\end{aligned}
$$

*with* $(\star)_{ij}$, $i,j=1,2$, *denoting the* $(i\,j)$*-element of* $\star$, $\mathbb{E}$ *defined in Proposition* 4.1, *and* $\vec{\mathbf{0}}:=(0,0,\dots,0)^{\mathrm{T}}$ $(\in\mathbb{R}^{N})$. *Set*

$$
\begin{aligned}
(\widehat{Q}^{o}_{0})_{11}&:=(\overset{o}{m}{}^{\infty}_{0})_{11}\Big(1+(\mathcal{R}^{o,\infty}_{0}(n))_{11}\Big)+(\mathcal{R}^{o,\infty}_{0}(n))_{12}(\overset{o}{m}{}^{\infty}_{0})_{21},\\
(\widehat{Q}^{o}_{0})_{12}&:=(\overset{o}{m}{}^{\infty}_{0})_{12}\Big(1+(\mathcal{R}^{o,\infty}_{0}(n))_{11}\Big)+(\mathcal{R}^{o,\infty}_{0}(n))_{12}(\overset{o}{m}{}^{\infty}_{0})_{22},\\
(\widehat{Q}^{o}_{0})_{21}&:=(\overset{o}{m}{}^{\infty}_{0})_{21}\Big(1+(\mathcal{R}^{o,\infty}_{0}(n))_{22}\Big)+(\mathcal{R}^{o,\infty}_{0}(n))_{21}(\overset{o}{m}{}^{\infty}_{0})_{11},\\
(\widehat{Q}^{o}_{0})_{22}&:=(\overset{o}{m}{}^{\infty}_{0})_{22}\Big(1+(\mathcal{R}^{o,\infty}_{0}(n))_{22}\Big)+(\mathcal{R}^{o,\infty}_{0}(n))_{21}(\overset{o}{m}{}^{\infty}_{0})_{12},\\
(\widehat{Q}^{o}_{1})_{11}&:=(\overset{o}{m}{}^{\infty}_{1})_{11}\Big(1+(\mathcal{R}^{o,\infty}_{0}(n))_{11}\Big)+(\mathcal{R}^{o,\infty}_{0}(n))_{12}(\overset{o}{m}{}^{\infty}_{1})_{21}+(\mathcal{R}^{o,\infty}_{1}(n))_{11}(\overset{o}{m}{}^{\infty}_{0})_{11}\\
&\quad+(\mathcal{R}^{o,\infty}_{1}(n))_{12}(\overset{o}{m}{}^{\infty}_{0})_{21},\\
(\widehat{Q}^{o}_{1})_{12}&:=(\overset{o}{m}{}^{\infty}_{1})_{12}\Big(1+(\mathcal{R}^{o,\infty}_{0}(n))_{11}\Big)+(\mathcal{R}^{o,\infty}_{0}(n))_{12}(\overset{o}{m}{}^{\infty}_{1})_{22}+(\mathcal{R}^{o,\infty}_{1}(n))_{11}(\overset{o}{m}{}^{\infty}_{0})_{12}\\
&\quad+(\mathcal{R}^{o,\infty}_{1}(n))_{12}(\overset{o}{m}{}^{\infty}_{0})_{22},\\
(\widehat{Q}^{o}_{1})_{21}&:=(\overset{o}{m}{}^{\infty}_{1})_{21}\Big(1+(\mathcal{R}^{o,\infty}_{0}(n))_{22}\Big)+(\mathcal{R}^{o,\infty}_{0}(n))_{21}(\overset{o}{m}{}^{\infty}_{1})_{11}+(\mathcal{R}^{o,\infty}_{1}(n))_{21}(\overset{o}{m}{}^{\infty}_{0})_{11}\\
&\quad+(\mathcal{R}^{o,\infty}_{1}(n))_{22}(\overset{o}{m}{}^{\infty}_{0})_{21},
\end{aligned}
$$

$$
\begin{aligned}
(\widehat{Q}_1^o)_{22} &:= (\overset{o}{m}{}_1^\infty)_{22}\Big(1+(\mathcal{R}_0^{o,\infty}(n))_{22}\Big)+(\mathcal{R}_0^{o,\infty}(n))_{21}(\overset{o}{m}{}_1^\infty)_{12}+(\mathcal{R}_1^{o,\infty}(n))_{21}(\overset{o}{m}{}_0^\infty)_{12}\\
&\quad+(\mathcal{R}_1^{o,\infty}(n))_{22}(\overset{o}{m}{}_0^\infty)_{22},\\
(\widehat{Q}_2^o)_{11} &:= (\overset{o}{m}{}_2^\infty)_{11}\Big(1+(\mathcal{R}_0^{o,\infty}(n))_{11}\Big)+(\mathcal{R}_0^{o,\infty}(n))_{12}(\overset{o}{m}{}_2^\infty)_{21}+(\mathcal{R}_1^{o,\infty}(n))_{11}(\overset{o}{m}{}_1^\infty)_{11}\\
&\quad+(\mathcal{R}_1^{o,\infty}(n))_{12}(\overset{o}{m}{}_1^\infty)_{21}+(\mathcal{R}_2^{o,\infty}(n))_{11}(\overset{o}{m}{}_0^\infty)_{11}+(\mathcal{R}_2^{o,\infty}(n))_{12}(\overset{o}{m}{}_0^\infty)_{21},\\
(\widehat{Q}_2^o)_{12} &:= (\overset{o}{m}{}_2^\infty)_{12}\Big(1+(\mathcal{R}_0^{o,\infty}(n))_{11}\Big)+(\mathcal{R}_0^{o,\infty}(n))_{12}(\overset{o}{m}{}_2^\infty)_{22}+(\mathcal{R}_1^{o,\infty}(n))_{11}(\overset{o}{m}{}_1^\infty)_{12}\\
&\quad+(\mathcal{R}_1^{o,\infty}(n))_{12}(\overset{o}{m}{}_1^\infty)_{22}+(\mathcal{R}_2^{o,\infty}(n))_{11}(\overset{o}{m}{}_0^\infty)_{12}+(\mathcal{R}_2^{o,\infty}(n))_{12}(\overset{o}{m}{}_0^\infty)_{22},\\
(\widehat{Q}_2^o)_{21} &:= (\overset{o}{m}{}_2^\infty)_{21}\Big(1+(\mathcal{R}_0^{o,\infty}(n))_{22}\Big)+(\mathcal{R}_0^{o,\infty}(n))_{21}(\overset{o}{m}{}_2^\infty)_{11}+(\mathcal{R}_1^{o,\infty}(n))_{21}(\overset{o}{m}{}_1^\infty)_{11}\\
&\quad+(\mathcal{R}_1^{o,\infty}(n))_{22}(\overset{o}{m}{}_1^\infty)_{21}+(\mathcal{R}_2^{o,\infty}(n))_{21}(\overset{o}{m}{}_0^\infty)_{11}+(\mathcal{R}_2^{o,\infty}(n))_{22}(\overset{o}{m}{}_0^\infty)_{21},\\
(\widehat{Q}_2^o)_{22} &:= (\overset{o}{m}{}_2^\infty)_{22}\Big(1+(\mathcal{R}_0^{o,\infty}(n))_{22}\Big)+(\mathcal{R}_0^{o,\infty}(n))_{21}(\overset{o}{m}{}_2^\infty)_{12}+(\mathcal{R}_1^{o,\infty}(n))_{21}(\overset{o}{m}{}_1^\infty)_{12}\\
&\quad+(\mathcal{R}_1^{o,\infty}(n))_{22}(\overset{o}{m}{}_1^\infty)_{22}+(\mathcal{R}_2^{o,\infty}(n))_{21}(\overset{o}{m}{}_0^\infty)_{12}+(\mathcal{R}_2^{o,\infty}(n))_{22}(\overset{o}{m}{}_0^\infty)_{22}.
\end{aligned}
$$

*Let* $\overset{o}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R}\to \mathrm{SL}_2(\mathbb{C})$ *be the solution of* **RHP2**. *Then,*

$$
\overset{o}{\mathrm{Y}}(z)z^{-(n+1)\sigma_3} \underset{z\to\infty}{=} \mathrm{Y}_0^{o,\infty}+\frac{1}{z}\mathrm{Y}_1^{o,\infty}+\frac{1}{z^2}\mathrm{Y}_2^{o,\infty}+\mathcal{O}(z^{-3}),
$$

*where*

$$
\begin{aligned}
(\mathrm{Y}_0^{o,\infty})_{11} &= (\widehat{Q}_0^o)_{11}\mathrm{e}^{-2(n+\frac{1}{2})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s},\\
(\mathrm{Y}_0^{o,\infty})_{12} &= (\widehat{Q}_0^o)_{12}\mathrm{e}^{n(\ell_o+2(1+\frac{1}{2n})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_0^{o,\infty})_{21} &= (\widehat{Q}_0^o)_{21}\mathrm{e}^{-n(\ell_o+2(1+\frac{1}{2n})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_0^{o,\infty})_{22} &= (\widehat{Q}_0^o)_{22}\mathrm{e}^{2(n+\frac{1}{2})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s},\\
(\mathrm{Y}_1^{o,\infty})_{11} &= \left((\widehat{Q}_1^o)_{11}-(2n+1)(\widehat{Q}_0^o)_{11}\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s\right)\mathrm{e}^{-2(n+\frac{1}{2})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s},\\
(\mathrm{Y}_1^{o,\infty})_{12} &= \left((\widehat{Q}_1^o)_{12}+(2n+1)(\widehat{Q}_0^o)_{12}\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s\right)\mathrm{e}^{n(\ell_o+2(1+\frac{1}{2n})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_1^{o,\infty})_{21} &= \left((\widehat{Q}_1^o)_{21}-(2n+1)(\widehat{Q}_0^o)_{21}\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s\right)\mathrm{e}^{-n(\ell_o+2(1+\frac{1}{2n})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_1^{o,\infty})_{22} &= \left((\widehat{Q}_1^o)_{22}+(2n+1)(\widehat{Q}_0^o)_{22}\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s\right)\mathrm{e}^{2(n+\frac{1}{2})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s},\\
(\mathrm{Y}_2^{o,\infty})_{11} &= \Bigg((\widehat{Q}_2^o)_{11}-(2n+1)(\widehat{Q}_1^o)_{11}\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s+(\widehat{Q}_0^o)_{11}\bigg(\tfrac{1}{2}(2n+1)^2\bigg(\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s\bigg)^2\\
&\quad-\tfrac{1}{2}(2n+1)\int_{J_o}s^2\psi_V^o(s)\,\mathrm{d}s\bigg)\Bigg)\mathrm{e}^{-2(n+\frac{1}{2})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s},\\
(\mathrm{Y}_2^{o,\infty})_{12} &= \Bigg((\widehat{Q}_2^o)_{12}+(2n+1)(\widehat{Q}_1^o)_{12}\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s+(\widehat{Q}_0^o)_{12}\bigg(\tfrac{1}{2}(2n+1)^2\bigg(\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s\bigg)^2\\
&\quad+\tfrac{1}{2}(2n+1)\int_{J_o}s^2\psi_V^o(s)\,\mathrm{d}s\bigg)\Bigg)\mathrm{e}^{n(\ell_o+2(1+\frac{1}{2n})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_2^{o,\infty})_{21} &= \Bigg((\widehat{Q}_2^o)_{21}-(2n+1)(\widehat{Q}_1^o)_{21}\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s+(\widehat{Q}_0^o)_{21}\bigg(\tfrac{1}{2}(2n+1)^2\bigg(\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s\bigg)^2\\
&\quad-\tfrac{1}{2}(2n+1)\int_{J_o}s^2\psi_V^o(s)\,\mathrm{d}s\bigg)\Bigg)\mathrm{e}^{-n(\ell_o+2(1+\frac{1}{2n})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s)},\\
(\mathrm{Y}_2^{o,\infty})_{22} &= \Bigg((\widehat{Q}_2^o)_{22}+(2n+1)(\widehat{Q}_1^o)_{22}\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s+(\widehat{Q}_0^o)_{22}\bigg(\tfrac{1}{2}(2n+1)^2\bigg(\int_{J_o}s\psi_V^o(s)\,\mathrm{d}s\bigg)^2
\end{aligned}
$$

$$+\ \tfrac{1}{2}(2n+1)\int_{J_o} s^2\psi_V^o(s)\,\mathrm{d}s\Bigg)\Bigg)\mathrm{e}^{2(n+\frac{1}{2})\int_{J_o}\ln(|s|)\psi_V^o(s)\,\mathrm{d}s}.$$

*Proof.* Let $\mathcal{R}^o\colon \mathbb{C}\setminus\widetilde{\Sigma}_p^o\to\mathrm{SL}_2(\mathbb{C})$ be the solution of the RHP $(\mathcal{R}^o(z),\upsilon_{\mathcal{R}}^o(z),\widetilde{\Sigma}_p^o)$ formulated in Proposition 5.2 with $n\to\infty$ asymptotics given in Lemma 5.3. For $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$, via the expansion $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, where $s\in\{b_{j-1}^o,a_j^o\}$, $j=1,\dots,N+1$, one obtains the asymptotics for $\mathcal{R}^o(z)$ stated in the Proposition.

Let $\overset{o}{m}{}^{\infty}\colon\mathbb{C}\setminus J_o^{\infty}\to\mathrm{SL}_2(\mathbb{C})$ solve the RHP $(\overset{o}{m}{}^{\infty}(z),J_o^{\infty},\overset{o}{\upsilon}{}^{\infty}(z))$ formulated in Lemma 4.3 with (unique) solution given by Lemma 4.5. In order to obtain large-$z$ asymptotics of $\overset{o}{m}{}^{\infty}(z)$, one needs large-$z$ asymptotics of $(\gamma^o(z))^{\pm1}$ and $\frac{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^o+\varepsilon_2\boldsymbol{d}_o)}{\boldsymbol{\theta}^o(\varepsilon_1\boldsymbol{u}^o(z)+\varepsilon_2\boldsymbol{d}_o)}$, $\varepsilon_1,\varepsilon_2=\pm1$. Consider, say, and without loss of generality, $z\to\infty$ asymptotics for $z\in\mathbb{C}_+$, that is, $z\to\infty^+$, where, by definition, $\sqrt{\star(z)}:=+\sqrt{\star(z)}$: equivalently, one may consider $z\to\infty$ asymptotics for $z\in\mathbb{C}_-$, that is, $z\to\infty^-$; however, recalling that $\sqrt{\star(z)}\restriction_{\mathbb{C}_+}=-\sqrt{\star(z)}\restriction_{\mathbb{C}_-}$, one obtains (in either case, and via the sheet-interchange index) the same $z\to\infty$ asymptotics (for $\overset{o}{m}{}^{\infty}(z)$). Recall the expression for $\gamma^o(z)$ given in Lemma 4.4: for $|z|\gg\max_{j=1,\dots,N+1}\{|b_{j-1}^o-a_j^o|\}$, via the expansions $\frac{1}{s-z}=-\sum_{k=0}^{l}\frac{s^k}{z^{k+1}}+\frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}_0^+$, and $\ln(z-s)=_{|z|\to\infty}\ln(z)-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{s}{z})^k$, where $s\in\{b_{j-1}^o,a_j^o\}$, $j=1,\dots,N+1$, one shows that, upon defining $\gamma^o(0)$ as in the Proposition,

$$(\gamma^o(z))^{\pm1}\underset{z\to\infty^+}{=}1+\frac{1}{z}\left(\pm\frac{1}{4}\sum_{k=1}^{N+1}(a_k^o-b_{k-1}^o)\right)+\frac{1}{z^2}\left(\pm\frac{1}{8}\sum_{k=1}^{N+1}\left((a_k^o)^2-(b_{k-1}^o)^2\right)\right.$$
$$\left.+\ \frac{1}{32}\left(\sum_{k=1}^{N+1}(a_k^o-b_{k-1}^o)\right)^2\right)+\mathcal{O}\left(\frac{1}{z^3}\right),$$

whence

$$\frac{1}{2}((\gamma_0^o)^{-1}\gamma^o(z)+\gamma_0^o(\gamma^o(z))^{-1})\underset{z\to\infty^+}{=}\frac{(\gamma_0^o+(\gamma_0^o)^{-1})}{2}+\frac{1}{z}\left(\left(\frac{\gamma_0^o-(\gamma_0^o)^{-1}}{8}\right)\sum_{k=1}^{N+1}\left(b_{k-1}^o-a_k^o\right)\right)$$
$$+\ \frac{1}{z^2}\left(\left(\frac{\gamma_0^o-(\gamma_0^o)^{-1}}{16}\right)\sum_{k=1}^{N+1}\left((b_{k-1}^o)^2-(a_k^o)^2\right)+\left(\frac{\gamma_0^o+(\gamma_0^o)^{-1}}{64}\right)\right.$$
$$\left.\times\left(\sum_{k=1}^{N+1}\left(b_{k-1}^o-a_k^o\right)\right)^2\right)+\mathcal{O}\left(\frac{1}{z^3}\right),$$

and

$$\frac{1}{2\mathrm{i}}((\gamma_0^o)^{-1}\gamma^o(z)-\gamma_0^o(\gamma^o(z))^{-1})\underset{z\to\infty^+}{=}-\frac{(\gamma_0^o-(\gamma_0^o)^{-1})}{2\mathrm{i}}+\frac{1}{z}\left(\left(\frac{\gamma_0^o+(\gamma_0^o)^{-1}}{8\mathrm{i}}\right)\sum_{k=1}^{N+1}\left(a_k^o-b_{k-1}^o\right)\right)$$
$$+\ \frac{1}{z^2}\left(\left(\frac{\gamma_0^o+(\gamma_0^o)^{-1}}{16\mathrm{i}}\right)\sum_{k=1}^{N+1}\left((a_k^o)^2-(b_{k-1}^o)^2\right)-\left(\frac{\gamma_0^o-(\gamma_0^o)^{-1}}{64\mathrm{i}}\right)\right.$$
$$\left.\times\left(\sum_{k=1}^{N+1}\left(a_k^o-b_{k-1}^o\right)\right)^2\right)+\mathcal{O}\left(\frac{1}{z^3}\right).$$

Recall from Lemma 4.5 that $\boldsymbol{u}^o(z):=\int_{a_{N+1}^o}^{z}\boldsymbol{\omega}^o$ ($\in\mathrm{Jac}(\mathcal{Y}_o)$, with $\mathcal{Y}_o:=\{(y,z);\,y^2=R_o(z)\}$), where $\boldsymbol{\omega}^o$, the associated normalised basis of holomorphic one-forms of $\mathcal{Y}_o$, is given by $\boldsymbol{\omega}^o=(\omega_1^o,\omega_2^o,\dots,\omega_N^o)$, with $\omega_j^o:=\sum_{k=1}^{N}c_{jk}^o(\prod_{i=1}^{N+1}(z-b_{i-1}^o)(z-a_i^o))^{-1/2}z^{N-k}\,\mathrm{d}z$, $j=1,\dots,N$, where $c_{jk}^o$, $j,k=1,\dots,N$, are obtained from Equations (O1) and (O2). Writing

$$\boldsymbol{u}^o(z)=\left(\int_{a_{N+1}^o}^{\infty^+}+\int_{\infty^+}^{z}\right)\boldsymbol{\omega}^o=\boldsymbol{u}_+^o(\infty)+\int_{\infty^+}^{z}\boldsymbol{\omega}^o,$$

where $\boldsymbol{u}^{o}_{+}(\infty) := \int_{a^{o}_{N+1}}^{\infty^{+}} \boldsymbol{\omega}^{o}$, for $|z| \gg \max_{j=1,\dots,N+1}\{|b^{o}_{j-1}-a^{o}_{j}|\}$, via the expansions $\frac{1}{s-z} = -\sum_{k=0}^{l} \frac{s^{k}}{z^{k+1}} + \frac{s^{l+1}}{z^{l+1}(s-z)}$, $l \in \mathbb{Z}^{+}_{0}$, and $\ln(z-s) =_{|z|\to\infty} \ln(z) - \sum_{k=1}^{\infty} \frac{1}{k}(\frac{s}{z})^{k}$, where $s \in \{b^{o}_{k-1}, a^{o}_{k}\}$, $k = 1, \dots, N+1$, one shows that, for $j = 1, \dots, N$,

$$\omega^{o}_{j} \underset{z\to\infty^{+}}{=} \frac{c^{o}_{j1}}{z^{2}}\,\mathrm{d}z + \frac{(c^{o}_{j2}+\frac{1}{2}c^{o}_{j1}\sum_{i=1}^{N+1}(a^{o}_{i}+b^{o}_{i-1}))}{z^{3}}\,\mathrm{d}z + \mathcal{O}\Big(\frac{\mathrm{d}z}{z^{4}}\Big),$$

whence

$$\begin{aligned}\int_{\infty^{+}}^{z} \omega^{o}_{j} &\underset{z\to\infty^{+}}{=} -\frac{c^{o}_{j1}}{z} - \frac{\frac{1}{2}(c^{o}_{j2}+\frac{1}{2}c^{o}_{j1}\sum_{i=1}^{N+1}(a^{o}_{i}+b^{o}_{i-1}))}{z^{2}} + \mathcal{O}\Big(\frac{1}{z^{3}}\Big)\\ &=: -\frac{\widehat{\alpha}^{o}_{\infty,j}}{z} - \frac{\widehat{\beta}^{o}_{\infty,j}}{z^{2}} + \mathcal{O}\Big(\frac{1}{z^{3}}\Big).\end{aligned}$$

Defining $\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})$, $\widetilde{\alpha}^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})$, and $\beta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})$, $\varepsilon_{1},\varepsilon_{2} = \pm 1$, as in the Proposition, recalling that $\boldsymbol{\omega}^{o} = (\omega^{o}_{1},\omega^{o}_{2},\dots,\omega^{o}_{N})$, and that the associated $N{\times}N$ Riemann matrix of $\boldsymbol{\beta}^{o}$-periods, $\tau^{o} = (\tau^{o})_{i,j=1,\dots,N} := (\oint_{\boldsymbol{\beta}^{o}_{j}} \omega^{o}_{i})_{i,j=1,\dots,N}$, is non-degenerate, symmetric, and $-\mathrm{i}\tau^{o}$ is positive definite, via the above asymptotic (as $z\to\infty^{+}$) expansion for $\int_{\infty^{+}}^{z}\omega^{o}_{j}$, $j=1,\dots,N$, one shows that

$$\frac{\boldsymbol{\theta}^{o}(\varepsilon_{1}\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_{2}\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\varepsilon_{1}\boldsymbol{u}^{o}(z)+\varepsilon_{2}\boldsymbol{d}_{o})} \underset{z\to\infty^{+}}{=} \Theta^{o}_{0} + \frac{1}{z}\Theta^{o}_{1} + \frac{1}{z^{2}}\Theta^{o}_{2} + \mathcal{O}\Big(\frac{1}{z^{3}}\Big),$$

where

$$\begin{aligned}\Theta^{o}_{0} &:= \frac{\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})}{\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})},\\ \Theta^{o}_{1} &:= \frac{\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})\widetilde{\alpha}^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}) - \widetilde{\alpha}^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})}{(\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}))^{2}},\\ \Theta^{o}_{2} &:= \Big(\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})\Big(\beta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}) + (\widetilde{\alpha}^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}))^{2}\Big) - \widetilde{\alpha}^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})\\ &\quad\times \widetilde{\alpha}^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}})\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}) - \beta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\boldsymbol{\Omega}^{o})(\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}))^{2}\Big)\,(\theta^{o}_{\infty}(\varepsilon_{1},\varepsilon_{2},\vec{\boldsymbol{0}}))^{-3},\end{aligned}$$

with $\vec{\boldsymbol{0}} := (0,0,\dots,0)^{\mathrm{T}}$ $(\in \mathbb{R}^{N})$. Via the above asymptotic (as $z\to\infty^{+}$) expansions for $\frac{1}{2}((\gamma^{o}_{0})^{-1}\gamma^{o}(z) + \gamma^{o}_{0}(\gamma^{o}(z))^{-1})$, $\frac{1}{2\mathrm{i}}((\gamma^{o}_{0})^{-1}\gamma^{o}(z) - \gamma^{o}_{0}(\gamma^{o}(z))^{-1})$, and $\frac{\boldsymbol{\theta}^{o}(\varepsilon_{1}\boldsymbol{u}^{o}(z)-\frac{1}{2\pi}(n+\frac{1}{2})\boldsymbol{\Omega}^{o}+\varepsilon_{2}\boldsymbol{d}_{o})}{\boldsymbol{\theta}^{o}(\varepsilon_{1}\boldsymbol{u}^{o}(z)+\varepsilon_{2}\boldsymbol{d}_{o})}$, one arrives at, upon recalling the expression for $\overset{o}{m}{}^{\infty}(z)$ given in Lemma 4.5, the asymptotic expansion for $\overset{o}{m}{}^{\infty}(z)$ stated in the Proposition.

Let $\overset{o}{\mathrm{Y}}\colon \mathbb{C}\setminus\mathbb{R} \to \mathrm{SL}_{2}(\mathbb{C})$ be the (unique) solution of **RHP2**, that is, $(\overset{o}{\mathrm{Y}}(z), \mathrm{I}+\exp(-n\widetilde{V}(z))\sigma_{+}, \mathbb{R})$. Recall, also, that, for $z\in\Upsilon^{o}_{1}$,

$$\overset{o}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_{o}}{2}\,\mathrm{ad}(\sigma_{3})}\mathcal{R}^{o}(z)\overset{o}{m}{}^{\infty}(z)\mathbb{E}^{\sigma_{3}}\mathrm{e}^{n(g^{o}(z)-\mathfrak{Q}^{+}_{\mathscr{A}})\sigma_{3}},$$

and, for $z\in\Upsilon^{o}_{2}$,

$$\overset{o}{\mathrm{Y}}(z) = \mathrm{e}^{\frac{n\ell_{o}}{2}\,\mathrm{ad}(\sigma_{3})}\mathcal{R}^{o}(z)\overset{o}{m}{}^{\infty}(z)\mathbb{E}^{-\sigma_{3}}\mathrm{e}^{n(g^{o}(z)-\mathfrak{Q}^{-}_{\mathscr{A}})\sigma_{3}}:$$

consider, say, and without loss of generality, large-$z$ asymptotics for $\overset{o}{\mathrm{Y}}(z)$ for $z\in\Upsilon^{o}_{1}$. Recalling from Lemma 3.4 that $g^{o}(z) := \int_{J_{o}} \ln((z-s)^{2+\frac{1}{n}}(zs)^{-1})\psi^{o}_{V}(s)\,\mathrm{d}s$, $z\in\mathbb{C}\setminus(-\infty,\max\{0,a^{o}_{N+1}\})$, for $|z| \gg \max_{j=1,\dots,N+1}\{|b^{o}_{j-1}-a^{o}_{j}|\}$, in particular, $|s/z| \ll 1$ with $s\in J_{o}$, and noting that $\int_{J_{o}}\psi^{o}_{V}(s)\,\mathrm{d}s = 1$ and $\int_{J_{o}} s^{m}\psi^{o}_{V}(s)\,\mathrm{d}s < \infty$, $m\in\mathbb{N}$, via the expansions $\frac{1}{s-z} = -\sum_{k=0}^{l}\frac{s^{k}}{z^{k+1}} + \frac{s^{l+1}}{z^{l+1}(s-z)}$, $l\in\mathbb{Z}^{+}_{0}$, and $\ln(z-s) =_{|z|\to\infty} \ln(z) - \sum_{k=1}^{\infty}\frac{1}{k}(\frac{s}{z})^{k}$, one shows that

$$\begin{aligned}g^{o}(z) \underset{z\to\infty}{=}& \Big(1+\frac{1}{n}\Big)\ln(z) - \int_{J_{o}}\ln(|s|)\psi^{o}_{V}(s)\,\mathrm{d}s - \mathrm{i}\pi\int_{J_{o}\cap\mathbb{R}_{-}}\psi^{o}_{V}(s)\,\mathrm{d}s + \frac{1}{z}\Big(-\Big(2+\frac{1}{n}\Big)\int_{J_{o}} s\psi^{o}_{V}(s)\,\mathrm{d}s\Big)\\ &+ \frac{1}{z^{2}}\Big(-\frac{1}{2}\Big(2+\frac{1}{n}\Big)\int_{J_{o}} s^{2}\psi^{o}_{V}(s)\,\mathrm{d}s\Big) + \mathcal{O}(z^{-3}),\end{aligned}$$

where $\int_{J_o \cap \mathbb{R}_-} \psi_V^o(s)\,\mathrm{d}s$ is given in Lemma 3.4. (Explicit expressions for $\int_{J_o} s^k \psi_V^o(s)\,\mathrm{d}s$, $k=1,2$, are given in Remark 3.2.) Using the asymptotic (as $z \to \infty$) expansions for $g^o(z)$, $\mathcal{R}^o(z)$, and $\overset{o}{m}{}^{\infty}(z)$ derived above, upon recalling the formula for $\overset{o}{\mathrm{Y}}(z)$, one arrives at, after a matrix-multiplication argument, the asymptotic expansion for $\overset{o}{\mathrm{Y}}(z) z^{-(n+1)\sigma_3}$ stated in the Proposition. $\square$

## Bibliography

[1] P. Deift and X. Zhou, "A Steepest Descent Method for Oscillatory Riemann-Hilbert Problems. Asymptotics for the MKdV Equation", Ann. of Math., Vol. 137, No. 2, pp. 295–368, 1993.

[2] P. Deift and X. Zhou, "Asymptotics for the Painlevé II Equation", Comm. Pure Appl. Math., Vol. 48, No. 3, pp. 277–337, 1995.

[3] P. Deift, S. Venakides, and X. Zhou, "New Results in Small Dispersion KdV by an Extension of the Steepest Descent Method for Riemann-Hilbert Problems", Int. Math. Res. Not., No. 6, pp. 285–299, 1997.

[4] W. B. Jones, W. J. Thron, and H. Waadeland, "A Strong Stieltjes Moment Problem", Trans. Amer. Math. Soc., Vol. 261, No. 2, pp. 503–528, 1980.

[5] W. B. Jones and W. J. Thron, "Survey of Continued Fractions Methods of Solving Moment Problems and Related Topics", pp. 4–37, in W. B. Jones, W. J. Thron, and H. Waadeland, eds., *Analytic Theory of Continued Fractions*, Lecture Notes in Mathematics, Vol. 932, Springer-Verlag, Berlin, 1982.

[6] W. B. Jones and W. J. Thron, "Orthogonal Laurent Polynomials and the Strong Hamburger Moment Problem", J. Math. Anal. Appl., Vol. 98, No. 2, pp. 528–554, 1984.

[7] O. Njåstad and W. J. Thron, "Unique Solvability of the Strong Hamburger Moment Problem", J. Austral. Math. Soc. Ser. A, Vol. 40, No. 1, pp. 5–19, 1986.

[8] O. Njåstad, "Solutions of the Strong Stieltjes Moment Problem", Methods Appl. Anal., Vol. 2, No. 3, pp. 320–347, 1995.

[9] O. Njåstad, "Extremal Solutions of the Strong Stieltjes Moment Problem", J. Comput. Appl. Math., Vol. 65, No. 1–3, pp. 309–318, 1995.

[10] O. Njåstad, "Solutions of the Strong Hamburger Moment Problem", J. Math. Anal. Appl., Vol. 197, No. 1, pp. 227–248, 1996.

[11] W. B. Jones and O. Njåstad, "Orthogonal Laurent Polynomials and Strong Moment Theory: A Survey", J. Comput. Appl. Math., Vol. 105, No. 1–2, pp. 51–91, 1999.

[12] T. J. Stieltjes, "Recherches sur les fractions continues", Ann. Fac. Sci. Univ. Toulouse, Vol. 8, J1–J122, 1894, Ann. Fac. Sci. Univ. Toulouse, Vol. 9, A1–A47, 1895 (in French).

[13] H. Hamburger, "Über eine Erweiterung des Stieltjesschen Momentproblems", Parts I, II, III, Math. Annalen, Vol. 81, pp. 235–319, 1920, Vol. 82, pp. 120–164, 1921, Vol. 82, pp. 168–187, 1921 (in German).

[14] G. Szegö, *Orthogonal Polynomials*, 4th edn., American Mathematical Society Colloquium Publications, Vol. 23, AMS, Providence, 1974.

[15] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, American Mathematical Society Surveys, Vol. II, AMS, New York, 1943.

[16] E. Hendriksen and H. van Rossum, "Orthogonal Laurent Polynomials", Nederl. Akad. Wetensch. Indag. Math., Vol. 48, No. 1, pp. 17–36, 1986.

[17] L. Cochran and S. C. Cooper, "Orthogonal Laurent Polynomials on the Real Line", pp. 47–100, in S. C. Cooper and W. J. Thron, eds., *Continued Fractions and Orthogonal Functions: Theory and Applications*, Lecture Notes in Pure and Applied Mathematics, Vol. 154, Marcel Dekker, Inc., New York, 1994.

[18] M. O. Cheney III, M. J. Witsoe, and S. C. Cooper, "A Comparison of Two Definitions for Orthogonal Laurent Polynomials", Commun. Anal. Theory Contin. Fract., Vol. 8, pp. 28–56, 2000.

[19] M. E. H. Ismail and D. R. Masson, "Generalized Orthogonality and Continued Fractions", J. Approx. Theory, Vol. 83, No. 1, pp. 1–40, 1995.

[20] A. Zhedanov, "The "Classical" Laurent Biorthogonal Polynomials", J. Comput. Appl. Math., Vol. 98, No. 1, pp. 121–147, 1998.

[21] A. Zhedanov, "Biorthogonal Rational Functions and the Generalized Eigenvalue Problem", J. Approx. Theory, Vol. 101, No. 2, pp. 303–329, 1999.

[22] E. Hendriksen, "The Strong Hamburger Moment Problem and Self-Adjoint Operators in Hilbert Space", J. Comput. Appl. Math., Vol. 19, No. 1, pp. 79–88, 1987.

[23] E. Hendriksen and C. Nijhuis, "Laurent-Jacobi Matrices and the Strong Hamburger Moment Problem", Acta Appl. Math., Vol. 61, No. 1–3, pp. 119–132, 2000.

[24] C. Díaz-Mendoza, P. González-Vera, and M. Jiménez Paiz, "Orthogonal Laurent Polynomials and Two-Point Padé Approximants Associated with Dawson's Integral", J. Comput. Appl. Math., Vol. 179, No. 1–2, pp. 195–213, 2005.

[25] A. Bultheel, P. Gonzalez-Vera, E. Hendriksen, and O. Njåstad, *Orthogonal Rational Functions*, Cambridge Monographs on Applied and Computational Mathematics, Vol. 5, Cambridge University Press, Cambridge, 1999.

[26] M. J. Cantero, L. Moral, and L. Velázquez, "Five-Diagonal Matrices and Zeros of Orthogonal Polynomials on the Unit Circle", Linear Algebra Appl., Vol. 362, pp. 29–56, 2003.

[27] B. Simon, "Orthogonal Polynomials on the Unit Circle: New Results", Int. Math. Res. Not., No. 53, pp. 2837–2880, 2004.

[28] B. Simon, *Orthogonal Polynomials on the Unit Circle. Part* 1*. Classical Theory*, American Mathematical Society Colloquium Publications, Vol. 54, Part 1, AMS, Providence, 2005.

[29] B. Simon, *Orthogonal Polynomials on the Unit Circle. Part* 2*. Spectral Theory*, American Mathematical Society Colloquium Publications, Vol. 54, Part 2, AMS, Providence, 2005.

[30] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njastad, "Orthogonal Rational Functions and Tridiagonal Matrices", J. Comput. Appl. Math., Vol. 153, No. 1–2, pp. 89–97, 2003.

[31] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad, "Quadrature and Orthogonal Rational Functions", J. Comput. Appl. Math., Vol. 127, No. 1–2, pp. 67–91, 2001.

[32] P. González-Vera and O. Njåstad, "Convergence of Two-Point Padé Approximants to Series of Stieltjes", J. Comput. Appl. Math., Vol. 32, No. 1–2, pp. 97–105, 1990.

[33] G. L. Lagomasino and A. M. Finkelshtein, "Rate of Convergence of Two-Point Padé Approximants and Logarithmic Asymptotics of Laurent-Type Orthogonal Polynomials", Constr. Approx., Vol. 11, No. 2, pp. 255–286, 1995.

[34] C. Díaz-Mendoza, P. González-Vera, and R. Orive, "On the Convergence of Two-Point Partial Padé Approximants for Meromorphic Functions of Stieltjes Type", Appl. Numer. Math., Vol. 53, No. 1, pp. 39–56, 2005.

[35] J. Coussement, A. B. J. Kuijlaars, and W. Van Assche, "Direct and Inverse Spectral Transform for the Relativistic Toda Lattice and the Connection with Laurent Orthogonal Polynomials", Inverse Problems, Vol. 18, No. 3, pp. 923–942, 2002.

[36] J. Coussement and W. Van Assche, "A Continuum Limit of the Relativistic Toda Lattice: Asymptotic Theory of Discrete Laurent Orthogonal Polynomials with Varying Recurrence Coefficients", J. Phys. A: Math. Gen., Vol. 38, No. 15, pp. 3337–3366, 2005.

[37] O. Njåstad, "A Modified Schur Algorithm and an Extended Hamburger Moment Problem", Trans. Amer. Math. Soc., Vol. 327, No. 1, pp. 283–311, 1991.

[38] K. T.-R. McLaughlin, A. H. Vartanian, and X. Zhou, "Asymptotics of Laurent Polynomials of Even Degree Orthogonal with Respect to Varying Exponential Weights", Preprint, 2006.

[39] K. T.-R. McLaughlin and P. D. Miller, "The $\overline{\partial}$ Steepest Descent Method and the Asymptotic Behavior of Polynomials Orthogonal on the Unit Circle with Fixed and Exponentially Varying Nonanalytic Weights", `arXiv:math.CA/0406484 v1`.

[40] K. T.-R. McLaughlin, A. H. Vartanian, and X. Zhou, "Asymptotics of Recurrence Relation Coefficients, Hankel Determinant Ratios, and Root Products Associated with Laurent Polynomials Orthogonal with Respect to Varying Exponential Weights", Preprint, 2006.

[41] A. S. Fokas, A. R. Its, and A. V. Kitaev, "Discrete Painlevé Equations and Their Appearance in Quantum Gravity", Comm. Math. Phys., Vol. 142, No. 2, pp. 313–344, 1991.

[42] A. S. Fokas, A. R. Its, and A. V. Kitaev, "The Isomonodromy Approach to Matrix Models in 2$D$ Quantum Gravity", Comm. Math. Phys., Vol. 147, No. 2, pp. 395–430, 1992.

[43] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Grundlehren der mathematischen Wissenschaften 316, Springer-Verlag, Berlin, 1997.

[44] P. Deift, T. Kriecherbauer, and K. T.-R. McLaughlin, "New Results on the Equilibrium Measure for Logarithmic Potentials in the Presence of an External Field", J. Approx. Theory, Vol. 95, No. 3, pp. 388–475, 1998.

[45] P. A. Deift, A. R. Its, and X. Zhou, "A Riemann-Hilbert Approach to Asymptotic Problems Arising in the Theory of Random Matrix Models, and also in the Theory of Integrable Statistical Mechanics", Ann. of Math., Vol. 146, No. 1, pp. 149–235, 1997.

[46] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, "Uniform Asymptotics for Polynomials Orthogonal with Respect to Varying Exponential Weights and Applications to Universality Questions in Random Matrix Theory", Comm. Pure Appl. Math., Vol. 52, No. 11, pp. 1335–1425, 1999.

[47] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, "Strong Asymptotics of Orthogonal Polynomials with Respect to Exponential Weights", Comm. Pure Appl. Math., Vol. 52, No. 12, pp. 1491–1552, 1999.

[48] J. Baik, P. Deift, and K. Johansson, "On the Distribution of the Length of the Longest Increasing Subsequence of Random Permutations", J. Amer. Math. Soc., Vol. 12, No. 4, pp. 1119–1178, 1999.

[49] T. Kriecherbauer and K. T.-R. McLaughlin, "Strong Asymptotics of Polynomials Orthogonal with Respect to Freud Weights", Int. Math. Res. Not., No. 6, pp. 299–333, 1999.

[50] S. Kamvissis, K. D. T.-R. McLaughlin, and P. D. Miller, *Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation*, Annals of Mathematics Studies, Vol. 154, Princeton University Press, New Jersey, 2003.

[51] P. D. Miller, "Asymptotics of Semiclassical Soliton Ensembles: Rigorous Justification of the WKB Approximation", Int. Math. Res. Not., No. 8, pp. 383–454, 2002.

[52] N. M. Ercolani and K. T.-R. McLaughlin, "Asymptotics of the Partition Function for Random Matrices via Riemann-Hilbert Techniques and Applications to Graphical Enumeration", Int. Math. Res. Not., No. 14, pp. 755–820, 2003.

[53] A. B. J. Kuijlaars, K. T.-R. McLaughlin, W. Van Assche, and M. Vanlessen, "The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1, 1]$", Adv. Math., Vol. 188, No. 2, pp. 337–398, 2004.

[54] A. I. Aptekarev, "Sharp Constants for Rational Approximations of Analytic Functions", Sb. Math., Vol. 193, No. 1, pp. 1–72, 2002.

[55] A. B. J. Kuijlaars, W. Van Assche, and F. Wielonsky, “Quadratic Hermite-Padé Approximation to the Exponential Function: a Riemann-Hilbert Approach”, Constr. Approx., Vol. 21, No. 3, pp. 351–412, 2005.

[56] P. Deift and X. Zhou, “Perturbation Theory for Infinite-Dimensional Integrable Systems on the Line. A Case Study.”, Acta Math., Vol. 188, No. 2, pp. 163–262, 2002 (Extended version: `http://www.ml.kva.se/publications/acta/webarticles/deift`).

[57] P. Deift and X. Zhou, “A Priori $L^p$-Estimates for Solutions of Riemann-Hilbert Problems”, Int. Math. Res. Not., No. 40, pp. 2121–2154, 2002.

[58] P. Deift and X. Zhou, “Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space”, Comm. Pure Appl. Math., Vol. 56, No. 8, pp. 1029–1077, 2003.

[59] A. I. Aptekarev and W. Van Assche, “Scalar and Matrix Riemann-Hilbert Approach to the Strong Asymptotics of Padé Approximants and Complex Orthogonal Polynomials with Varying Weights”, J. Approx. Theory, Vol. 129, No. 2, pp. 129–166, 2004.

[60] A. B. J. Kuijlaars and M. Vanlessen, “Universality for Eigenvalue Correlations at the Origin of the Spectrum”, Comm. Math. Phys., Vol. 243, No. 1, pp. 163–191, 2003.

[61] M. Vanlessen, “Strong Asymptotics of the Recurrence Coefficients of Orthogonal Polynomials Associated to the Generalized Jacobi Weight”, J. Approx. Theory, Vol. 125, No. 2, pp. 199–229, 2003.

[62] J. Baik, T. Kriecherbauer, K. T.-R. McLaughlin, and P. D. Miller, “Uniform Asymptotics for Polynomials Orthogonal with Respect to a General Class of Discrete Weights and Universality Results for Associated Ensembles”, `arXiv:math.CA/0310278 v1`.

[63] A. B. J. Kuijlaars and A. Martínez-Finkelshtein, “Strong Asymptotics for Jacobi Polynomials with Varying Nonstandard Parameters”, J. Anal. Math., Vol. 94, pp. 195–234, 2004.

[64] P. Deift and D. Gioev, ‘Universality in Random Matrix Theory for Orthogonal and Symplectic Ensembles”, `arXiv:math-ph/0411075 v3`.

[65] P. Deift and D. Gioev, ‘Universality at the Edge of the Spectrum for Unitary, Orthogonal and Symplectic Ensembles of Random Matrices”, `arXiv:math-ph/0507023 v2`.

[66] G. Lyng and P. D. Miller, “The $N$-Soliton of the Focusing Nonlinear Schrödinger Equation for $N$ Large”, `arXiv:nlin.SI/0508007 v1`.

[67] I. V. Krasovsky, “Gap Probability in the Spectrum of Random Matrices and Asymptotics of Polynomials Orthogonal on an Arc of the Unit Circle”, Int. Math. Res. Not., No. 25, pp. 1249–1272, 2004.

[68] T. Claeys, A. B. J. Kuijlaars, and M. Vanlessen, “Multi-Critical Unitary Random Matrix Ensembles and the General Painlevé II Equation”, `arXiv:math-ph/0508062 v1`.

[69] M. Vanlessen, “Strong Asymptotics of Laguerre-Type Orthogonal Polynomials and Applications in Random Matrix Theory”, `arXiv:math.CA/0504604 v2`.

[70] M. B. Hernández and A. M. Finkelshtein, “Zero Asymptotics of Laurent Orthogonal Polynomials”, J. Approx. Theory, Vol. 85, No. 3, pp. 324–342, 1996.

[71] D. K. Dimitrov and A. Sri Ranga, “Monotonicity of Zeros of Orthogonal Laurent Polynomials”, Methods Appl. Anal., Vol. 9, No. 1, pp. 1–12, 2002.

[72] A. Sri Ranga and W. Van Assche, “Blumenthal’s Theorem for Laurent Orthogonal Polynomials”, J. Approx. Theory, Vol. 117, No. 2, pp. 255–278, 2002.

[73] A. Sri Ranga, “On a Recurrence Formula Associated with Strong Distributions”, SIAM J. Math. Anal., Vol. 21, No. 5, pp. 1335–1348, 1990.

[74] R. Beals and R. R. Coifman, "Scattering and Inverse Scattering for First Order Systems", Comm. Pure Appl. Math., Vol. 37, No. 1, pp. 39–90, 1984.

[75] X. Zhou, "Direct and Inverse Scattering Transforms with Arbitrary Spectral Singularities", Comm. Pure Appl. Math., Vol. 42, No. 7, pp. 895–938, 1989.

[76] X. Zhou, "Inverse Scattering Transform for Systems with Rational Spectral Dependence", J. Differential Equations, Vol. 115, No. 2, pp. 277–303, 1995.

[77] G. Springer, *Introduction to Riemann Surfaces*, 2nd edn., Chelsea, New York, 1981.

[78] H. M. Farkas and I. Kra, *Riemann Surfaces*, 2nd edn., Graduate Texts in Mathematics, Vol. 71, Springer-Verlag, New York, 1992.

[79] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Courant Lecture Notes in Mathematics, Vol. 3, Courant Institute of Mathematical Sciences, New York, 1999.

[80] K. Johansson, "On Fluctuations of Eigenvalues of Random Hermitian Matrices", Duke Math. J., Vol. 91, No. 1, pp. 151–204, 1998.

[81] A. B. J. Kuijlaars and K. T.-R. McLaughlin, "Generic Behavior of the Density of States in Random Matrix Theory and Equilibrium Problems in the Presence of Real Analytic External Fields", Comm. Pure Appl. Math., Vol. 53, No. 6, pp. 736–785, 2000.

[82] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1972.

[83] F. D. Gakhov, *Boundary Value Problems*, Dover Publications, Inc., New York, 1990.

[84] P. Deift and K. T.-R. McLaughlin, *A Continuum Limit of the Toda Lattice*, Mem. Amer. Math. Soc., Vol. 131, No. 624, AMS, Providence, 1998.

[85] S. B. Damelin, P. D. Dragnev, and A. B. J. Kuijlaars, "The Support of the Equilibrium Measure for a Class of External Fields on a Finite Interval", Pacific J. Math., Vol. 199, No. 2, pp. 303–320, 2001.

[86] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*, Operator Theory: Advances and Applications, Vol. 3, Birkhäuser, Basel, 1981.

[87] R. Beals, P. Deift, and C. Tomei, *Direct and Inverse Scattering on the Line*, Mathematical Surveys and Monographs, No. 28, AMS, Providence, 1988.

[88] P. Deift and X. Zhou, "Direct and Inverse Scattering on the Line with Arbitrary Singularities", Comm. Pure Appl. Math., Vol. 44, No. 5, pp. 485–533, 1991.

[89] X. Zhou, "The Riemann-Hilbert Problem and Inverse Scattering", SIAM J. Math. Anal., Vol. 20, No. 4, pp. 966–986, 1989.

[90] X. Zhou, "Strong Regularizing Effect of Integrable Systems", Comm. Partial Differential Equations, Vol. 22, Nos. 3 & 4, pp. 503–526, 1997.

[91] X. Zhou, "$L^2$-Sobolev Space Bijectivity of the Scattering and Inverse Scattering Transforms", Comm. Pure Appl. Math., Vol. 51, No. 7, pp. 697–731, 1998.

[92] J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Vol. 68, Springer-Verlag, New York, 1980.